Assignment 1: ODE review
Due Wednesday January 28, 2026

John McCuan

In the following problems, let L denote a positive real number.

Problem 1 Use "elementary methods" to find all twice differentiable functions $x = x(t)$ satisfying

$$x'' + x' - 6x = t(L - t)$$

where L is a positive real number. Write your solution in the form

$$x(t) = x_h(t) + x_p(t)$$

where $x_p(t) = t^2/6 + (1 - 3L)t/18 + (7 - 3L)/108$ gives a particular solution.

Problem 2 Solve the initial value problem

$$\begin{cases} x'' + x' - 6x = t(L - t) \\ x(0) = 0 = x'(0) \end{cases} \tag{1}$$

using "elementary methods."

Problem 3 (Haberman 13.2.1) Recall that the Laplace transform of a locally integrable function $f : [0, \infty) \to \mathbb{R}$ is defined by

$$\mathcal{L}[f] = \int_0^\infty e^{-st} f(t)\, dt.$$

Using explicit integration make for yourself a short table giving the Laplace transforms of the functions determined by the following values:

(a) $f(t) \equiv 1$.

(b) $f(t) = t$.

(c) $f(t) = t^2$.

(d) $f(t) = t(L - t)$.

(e) $f(t) = e^{\alpha t}$

Problem 4 (Haberman 13.2) Use explicit integration to verify the derivative formulas:

(a) $\mathcal{L}[x'] = -x(0) + s\mathcal{L}[x]$ and

(b) $\mathcal{L}[x''] = -x'(0) - sx(0) + s^2\mathcal{L}[x]$

for $x \in C^2[0, \infty)$.

Problem 5 (Haberman 10.3.14) The gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ has value given by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt.$$

Show the following:

(a) $\Gamma(1) = 1$.

(b) $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

(c) $\Gamma(n + 1) = n!$ for $n = 1, 2, 3, \ldots$.

(d) $\Gamma(1/2) = \sqrt{\pi}$. Hint: Use the substitution $u = \sqrt{t}$ and evaluate $(\Gamma(1/2))^2$ as a double integral in polar coordinates.

Problem 6 (Haberman 13.2.2) Show

$$\mathcal{L}[t^\alpha] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

for $\alpha > -1$.

Problem 7 Find the Laplace transform of the IVP (1) given in Problem 2, and solve this (algebraic) problem for $\mathcal{L}[x]$ as a function of s.

Problem 8 Recall the partial fractions decomposition

$$\frac{Ls-2}{s^3(s+3)(s-2)} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s^3} + \frac{d}{s+3} + \frac{e}{s-2}. \tag{2}$$

Solve for the constants a, b, c, d and e to give an alternative derivation of the solution you found in Problem 2 for the IVP (1).

Problem 9 (Haberman 13.2 and 13.3) The convolution theorem for the Laplace transform asserts that if f and g are locally integrable functions on the interval $[0,\infty)$, then the function $h : [0,\infty) \to \mathbb{R}$ with values given by the convolution integral

$$h(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

is locally integrable and the Laplace transform of h is given by the product

$$\mathcal{L}[h] = \mathcal{L}[f]\mathcal{L}[g].$$

Use the convolution theorem to find the inverse Laplace transform of the product

$$\frac{Ls-2}{s^3(s+3)(s-2)} = \frac{Ls-2}{s^3}\;\frac{1}{(s+3)(s-2)}$$

in (2) without solving the big system of partial fractions. Hint: Use a much simpler partial fraction decomposition to find a function g with

$$\mathcal{L}[g] = \frac{1}{(s+3)(s-2)}.$$

This gives an alternative to the derivation of Problem 7 and a second alternative derivation using the Laplace transform of the solution to the IVP in Problem 2.

Problem 10 (two point boundary value problem) Consider the two point boundary value problem

$$\begin{cases} x'' + tx = t(L-t) \\ x(0) = 0 = x(L). \end{cases} \tag{3}$$

(a) Find a particular solution x_p for the ODE. Hint: Take the very simple one with $x_p'' \equiv 0$.

(b) Formulate an appropriate two point boundary value problem for the associated homogeneous solution x_h.

(c) Solve the homogeneous ODE in the problem you formulated in part **(b)** using a power series expansion at $t = 0$ to obtain a solution of the form

$$x_h(t) = a_0 X_0(t) + a_1 X_1(t)$$

with

$$\begin{cases} X_0(t) &= \sum_{j=0}^{\infty} \alpha_j t^j, \text{ and} \\ \\ X_1(t) &= \sum_{j=0}^{\infty} \beta_j t^j. \end{cases}$$

(d) Use the boundary values for $x_h(0)$ and $x_h(L)$ from part **(b)** and numerics to determine when and if you can expect to solve this problem.