Assignment 1: ODE
Pace: Thursday August 29, 2024, Due Tuesday September 3, 2024

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In the problems below we refer to the following results:

Theorem 1 (general local existence and uniqueness) If

$$\mathbf{F} \in C^1(\mathbb{R}^n \times (a,b) \to \mathbb{R}^n),$$

then for any $\mathbf{p} \in \mathbb{R}^n$ and any $t_0 \in (a,b)$ there exists some $\epsilon > 0$ such that the initial value problem (IVP)

$$\begin{cases} \mathbf{x}' = \mathbf{F}(\mathbf{x}, t) & t_0 - \epsilon < t < t_0 + \epsilon \\ \mathbf{x}(t_0) = \mathbf{p} \end{cases} \tag{1}$$

has a unique solution.

Theorem 2 (existence and uniqueness theorem for linear ODE) Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$. If $a_{ij}, b_j \in C^0(a,b)$ for $i, j = 1, 2, \ldots, n$, then for every $(\mathbf{p}, t_0) \in \mathbb{R}^n \times (a,b)$ the IVP

$$\begin{cases} \mathbf{x}' = A\mathbf{x} + \mathbf{b}, & t \in (a,b) \\ \mathbf{x}(t_0) = \mathbf{p}, \end{cases} \tag{2}$$

where $A \in C^0((a,b) \to \mathbb{R}^{n \times n})$ is the $n \times n$ matrix malued function with the real valued function a_{ij} in the i-th row and j-th column and $\mathbf{b} \in C^0((a,b) \to \mathbb{R}^n)$ is the vector valued function with j-th component function b_j, has a unique solution $\mathbf{x} \in C^1((a,b) \to \mathbb{R}^n)$.

Problem 1 (continuity) Let a and b be extended real numbers in $\mathbb{R} \cup \{\pm\infty\}$ with $a < b$, and let U be an open subset of $\mathbb{R}^n$ for some $n \in \mathbb{N} = \{1, 2, \ldots\}$ (the natural numbers).

(a) State carefully the definition of continuity for a function $f : (a, b) \to \mathbb{R}$.

(b) If $f, g \in C^0(U)$, show $f + g \in C^0(U)$.

(c) If $f \in C^0(U)$, show $cf \in C^0(U)$ for every $c \in \mathbb{R}$.

These are the two main properties making $C^0(U)$ a **vector space**.

Problem 2 (initial value problem) If $\mathbf{x}$ is the solution of the initial value problem in the general existence and uniqueness theorem for ODEs, then it is natural to assume $\mathbf{x}$ is **differentiable**. Show that in fact, under the assumptions of the theorem the solution is **continuously differentiable** that is $\mathbf{x} \in C^1((t_0 - \epsilon, t_0 + \epsilon) \to \mathbb{R}^n)$.

Problem 3 (an ODE) Solve the IVP:

$$\begin{cases} \mathbf{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x} \\ \\ \mathbf{x}(t_0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \end{cases}$$

Problem 4 (IVP) Consider the initial value problem

$$\begin{cases} y'' = y^2 \\ y(0) = 3. \end{cases} \tag{3}$$

(a) What does Theorem 1 tell you about the solutions to this problem?

(b) Explore the following assertion numerically:

> There exists a uniform $\epsilon > 0$ for which all solutions of (3) are well-defined and unique on the interval $(-\epsilon, \epsilon)$, that is for $-\epsilon < x < \epsilon$.

(c) Multiply both sides of the ODE by y' and integrate to obtain an implicit solution. Does this tell you anything decisive about the assertion of part **(b)**?

Problem 5 (regularity) Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$. Show that if $f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$, then f is continuous at $x_0 \in (a, b)$.

Problem 6 (open set) A set $U \subset \mathbb{R}^n$ is said to be an **open set** if for each $\mathbf{p} \in U$ there exists some $r > 0$ so that

$$B_r(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\} \subset U. \tag{4}$$

The set $B_r(\mathbf{p})$ defined in (4) is called an **open ball**.

(a) Show that an open interval (a, b) is an open set in $\mathbb{R}^1$.

(b) Show that an open ball is an open set.

(c) Show that the intersection

$$\bigcap_{j=1}^{k} U_j = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \in U_j,\ \ j = 1, 2, \ldots, k\}$$

where $U_1, U_2, \ldots, U_k$ are open sets in $\mathbb{R}^n$ is an open set in $\mathbb{R}^n$, i.e., any intersection of finitely many open sets is an open set.

(d) Show that the intersection of infinitely many open sets need not be an open set.

Problem 7 (uniqueness) Consider the IVP

$$\begin{cases} y' = \sqrt{|y|}, & x \in \mathbb{R} \\ y(0) = 0 \end{cases} \tag{5}$$

and the function $y_1 : \mathbb{R} \to \mathbb{R}$ given by

$$y_1(x) = \begin{cases} (1/4)x^3/|x|, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

(a) What does Theorem 1 tell you about the solution(s) of (5)? In particular:

(i) In order to apply Theorem 1 to the IVP (5), identify an appropriate function $f : \mathbb{R} \to \mathbb{R}$ for which $y' = f$.

(ii) Does f satisfy the hypothesis of the theorem in the case $n = 1$? Why or why not?

(b) Show $y_1 \in C^1(\mathbb{R})$. In particular:

(i) Draw the graphs of y_1 and y_1'; properly label the axes.

(ii) Draw the graph of f from part **(a)** above; properly label the axes.

(c) Show y_1 satisfies (5).

(d) Find three other solutions of (5) and draw the graphs of two of the solutions you find.

Problem 8 (another IVP) Consider the IVP

$$\begin{cases} y' = y^2, \\ y(t_0) = y_0. \end{cases} \tag{6}$$

(a) What does Theorem 2 tell you about the solution(s) of (6)?

(b) Solve (6). Hint: The ODE is separable.

(c) For each $(y_0, t_0) \in \mathbb{R}^2$ there exists a unique smallest extended real number $a \in [-\infty, t_0)$ and a unique largest real number $b \in (t_0, \infty]$ for which (6) has a unique solution on the interval (a, b). Find a and b (as functions of t_0 and y_0).

Problem 9 (linear IVP) Consider the IVP

$$\begin{cases} y'' = y, \\ y(1) = 1. \end{cases} \tag{7}$$

(a) What does Theorem 2 tell you about the solution(s) of (7)? In particular:

(i) In order to apply Theorem 2 to the IVP (7), identify an appropriate matrix valued function $A : \mathbb{R} \to \mathbb{R}^{2\times 2}$ for which the ODE in (7) is equivalent to $\mathbf{x}' = A\mathbf{x}$.

(ii) Identify the initial point $\mathbf{p}$ for application of Theorem 2.

(b) Solve the IVP for the system you identified in part **(a)** above.

(c) Solve (7).

(d) Plot at least three different solutions of (7). You may wish to use mathematical software like Matlab, Maple, or Mathematica. Why does this not violate the uniqueness assertion of Theorems 1 and 2?

Problem 10 (two point boundary value problem) Given $L > 0$, a function $f \in C^0[0, L]$, and $c, d \in \mathbb{R}$ consider the BVP

$$\begin{cases} y'' = f(x), \quad x \in (0, L) \\ y(0) = c, \\ y(L) = d. \end{cases} \tag{8}$$

Find a function $g \in C^0[0, L]$ so that the BVP (8) is **equivalent** to the BVP

$$\begin{cases} u'' = g(x), \quad x \in (0, L) \\ u(0) = 0, \\ u(L) = 0. \end{cases} \tag{9}$$

Once you find g, complete the following:

(a) Given the unique solution u of (9) you can find a formula for the unique solution y of (8) in terms of u.

(b) Given the unique solution y of (8) you can find a formula for the unique solution u of (9) in terms of y.

Hint: The function g should depend on f, c and d.