Assignment 1: The 1-D Heat Equation (Intro)
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Problem 1 *(Haberman 1.2.1-3)*

(a) *Determine the dimensions of* **lineal heat energy density** $\theta = \theta(x, t)$ *for which*

$$\int_a^b \theta(x, t)\, dx$$

models the total heat energy in a thin rod found between positions $x = a$ *and* $x = b$.

(b) *Consider a (more complicated) thin rod with varying cross-sectional area* $A = A(x)$ *and* **volumetric heat energy density** *also called* $\theta = \theta(x, t)$. *Determine the expression for the total heat energy between positions* $x = a$ *and* $x = b$.

(c) *Derive a heat equation for the temperature* $u = u(x, t)$ *in the rod of varying cross-sectional area from the previous part under the assumption of constant specific heat capacity* c, *constant volumetric density* ρ, *and constant heat conductivity* K.

Problem 2 *(Haberman 1.2.8) Give an expression for the total thermal energy in a rod modeled on an interval* $0 \le x \le L$ *in terms of the temperature* $u = u(x, t)$.

Problem 3 *If* f *is a continuous function defined on the interval* $[a, b]$ *and*

$$\int_x^{x+\Delta x} f(\xi)\, d\xi = 0 \qquad \text{whenver } a < x < x + \Delta x < b$$

then explain clearly and carefully why $f((a + b)/2) = 0$.

Problem 4 *Find a solution* $u = u(x, t)$ *of the problem*

$$\begin{cases} u_t = u_{xx}, & 0 < x < L \\ u(0,t) = u_0, & t > 0 \\ u(L,t) = u_1, & t > 0 \end{cases}$$

where u_0 *and* u_1 *are given constants.*

Problem 5 *Find as many solutions* $u = u(x, t)$ *as you can to the problem*

$$\begin{cases} u_t = u_{xx}, & 0 < x < L \\ u_x(0,t) = 1 = u_x(L,t), & t > 0. \end{cases}$$

Do you think you have found all solutions?

Problem 6 *Consider functions* $u = u(x, t)$ *of the form*

$$u(x,t) = \frac{1}{L}[u_1(t) - u_0(t)]\, x + u_0(t).$$

(a) *What can you say about* u_0 *and* u_1 *if* u *is a solution of* $u_t = u_{xx}$ *?*

(b) *What if* $u_1(t) - u_0(t)$ *is a constant but* $u_0(t)$ *is not a constant?*

(c) *Give an example where the conditions in the previous part hold for* u_0 *and* u_1.

Problem 7 *Let* u_0 *and* u_1 *be constants. Consider the problem*

$$\begin{cases} u_t = u_{xx}, & 0 < x < L \\ u(0,t) = u_0, & t > 0 \\ u(L,t) = u_1, & t > 0 \\ u(x,0) = g(x), & 0 \le x \le L \end{cases}$$

where $g = g(x)$ *is a given function. Find the equilibrium solution* $u_* = u_*(x)$ *associated with this problem and formulate an (interesting) assertion/guess about the relation between the solution* $u = u(x, t)$ *and the equilibrium solution* $u_*(x)$.

Problem 8 *(Haberman 1.4.1) Find the equilibrium solution associated with the problem*

$$\begin{cases} u_t = u_{xx} + x^2, & 0 < x < L \\ u(0,t) = u_0, & t > 0 \\ u_x(L,t) = 0, & t > 0 \\ u(x,0) = g(x), & 0 \le x \le L. \end{cases}$$

Here u_0 *is a given constant and* g *is a given function.*

Problem 9 *(Haberman 1.4.2) Consider an* **equilibrium/steady state** *solution* u_* *of the one-dimensional heat equation on the interval* $0 \leq x \leq L$ *with constant conductivity* K*, fixed boundary temperatures* $u_*(0) = 0 = u_*(L)$*, and internal thermal energy rate-density generation/forcing modeled by* $Q(x) = x$*.*

(a) *Find an expression for the heat energy generated per unit time along the entire rod.*

(b) *Find an expression for the rate of heat energy flowing out of the rod at the ends* $x = 0$ *and at* $x = L$ *(in terms of* u_**).*

(c) *What relation should hold between your answers to the first two parts?*

Problem 10 *(Haberman 1.4.3) Determine the equilibrium temperature distribution for a one-dimensional rod consisting of two different materials in perfect thermal contact at* $x = 1$ *and satisfying the following conditions:*

(i) *The material modeled on* $0 \leq x < 1$ *has* $c\rho = 1$ *and* $K = 1$ *(where* c *is the specific heat capacity,* ρ *is the density, and* K *is the conductivity). Also on* $0 \leq x < 1$ *there is an internal unit heat source with constant density per time given by* $Q = 1$*.*

(ii) *The material modeled on* $1 < x \leq 2$ *has* $c\rho = 2$ *and* $K = 2$ *and* $Q = 0$*.*

(iii) $u(0) = 0 = u(2)$*.*

Perfect thermal contact *means the temperature* $u = u(x)$ *is continuous at* $x = 1$ *and the thermal energy exiting the portion of the rod modeled by* $0 < x < 1$ *is equal to the thermal energy entering the portion of the rod modeled by* $1 < x < 2$*. Be careful: This does not mean* $u_*'(1^-) = u_*'(1^+)$*. You need to use Fourier's law. See also Haberman 1.3.2.*