Math 4431, Exam 2: Ch. 3 (practice) Name and section: ____________________

1. (a) (10 points) Give a precise definition of what it means for a topological space X to be **compact**.

   (b) (10 points) Let $K \subset X$ be compact (in the subspace topology). If $\{U_\alpha\}_{\alpha\in\Gamma}$ is a family of open sets in X such that

$$K \subset \cup_{\alpha\in\Gamma} U_\alpha,$$

then show there is a finite subfamily $\{U_{\alpha_1}, \ldots, U_{\alpha_k}\}$ such that

$$K \subset \cup_{j=1}^{k} U_{\alpha_j}.$$

Solution:

(a) X is compact if for every family $\{U_\alpha\}_{\alpha\in\Gamma}$ of open sets in X such that

$$X = \cup_{\alpha\in\Gamma} U_\alpha,$$

there is a finite subfamily $\{U_{\alpha_1}, \ldots, U_{\alpha_k}\}$ such that

$$X = \cup_{j=1}^{k} U_{\alpha_j}.$$

(b) If $\{U_\alpha\}_{\alpha\in\Gamma}$ is an open cover of K by open sets in X, then $\{U_\alpha \cap K\}_{\alpha\in\Gamma}$ is an open cover of K by open sets in K. By compactness, there is a finite subcover

$$K = \cup_{j=1}^{k} (U_{\alpha_j} \cap K),$$

and clearly

$$K \subset \cup_{j=1}^{k} U_{\alpha_j}.$$

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2. (20 points) Prove that the continuous image of a compact space is compact.

Solution: Let X be a compact space and $f : X \to Y$ a continuous function. If $f(X) \subset \cup_{\alpha\in\Gamma} V_\alpha$ for some open sets V_α in Y, then by continuity

$$\{f^{-1}(V_\alpha)\}_{\alpha\in\Gamma}$$

is an open cover of X. By the compactness of X, there is a finite subcover

$$\{f^{-1}(V_{\alpha_1}), \dots, f^{-1}(V_{\alpha_k})\}$$

of X. If $y = f(x) \in f(X)$, then there is some j for which $x \in f^{-1}(V_{\alpha_j})$. This means, $y = f(x) \in V_{\alpha_j}$. Therefore, $\{V_{\alpha_1}, \dots, V_{\alpha_k}\}$ is a finite subcover of $f(X)$. Therefore, $f(X)$ is compact.

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3. (3.4.25) Let X be a topological space, and consider the function $f : X \to X \times X$ by $f(x) = (x, x)$.

   (a) (10 points) Show that f is continuous.

   (b) (10 points) Show that if the diagonal $f(X)$ is closed in $X \times X$, then X is Hausdorff.

Solution:

(a) It is enough to show that the inverse image of a basic open set $U \times V$ where U and V are open in X is open. In fact,

$$f^{-1}(U \times V) = U \cap V,$$

so f is continuous.

(b) Let $x_1 \neq x_2$ be points in X and denote the diagonal by $\Delta = f(X)$. Then (x_1, x_2) is in the open set $X \times X \backslash \Delta$. There is a basic open set $U \times V$ with $(x_1, x_2) \in U \times V \subset \Delta^c$. That is, $x_1 \in U$, $x_2 \in V$, and $(\xi_1, \xi_2) \in U \times V$ implies $\xi_1 \neq \xi_2$. This implies $U \cap V = \phi$ and X is Hausdorff since if $\xi \in U \cap V$, then $(\xi, \xi) \in U \times V$.

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4. (a) (10 points) Give the precise definition of what it means for a topological space X to be **connected**.

   (b) (10 points) Prove that if A and B are connected subspaces of a topological space X and $A \cap B \neq \phi$, then $A \cup B$ is connected.

Solution:

(a) X is **connected** if whenever U_1 and U_2 are disjoint open sets with $U_1 \cup U_2 = X$, then either $U_1 = \phi$ or $U_2 = \phi$. (Sometimes it may also be required for convenience that $X \neq \phi$.)

(b) Assume $A \cup B = U_1 \cup U_2$ for disjoint open sets U_j, $j = 1, 2$. Let $x \in A \cap B$. The element x is in exactly one of U_1 or U_2, but not both (since they are disjoint). Say $x \in U_1$. If $U_2 \cap A \neq \phi$, then $V_1 = U_1 \cap A$ and $V_2 = U_2 \cap A$ are disjoint nonempty open sets in A with $A = V_1 \cup V_2$. This contradicts the hypothesis that A is connected. If, on the other hand, $U_2 \cap A = \phi$, then $U_2 \subset B$, and $W_1 = U_1 \cap B$ and $W_2 = U_2 \cap B$ are disjoint open sets with $B = W_1 \cup W_2$. Since B is connected and $W_1 \neq \phi$, we must have $W_2 = U_2 \cap B = \phi$. But in this case $U_2 \subset B$, so $U_2 \cap B = U_2 = \phi$.

The case $x \in U_2$ can be considered symmetrically leading to the conclusion $U_1 = \phi$. This shows $A \cup B$ is connected.

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5. (a) (10 points) Define the **product space** $X \times Y$ of two topological spaces.
   (b) (10 points) Prove that if X and Y are path connected spaces, then $X \times Y$ is connected.

Solution:

(a) $X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$ is the topological space with basis

$$\mathcal{B} = \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}.$$

(b) Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Let $\gamma : [0,1] \to X$ be a (continuous) path connecting x_1 to x_2 in X. Let $\eta : [0,1] \to X$ be a (continuous) path connecting y_1 to y_2 in Y.

Consider the function $\phi : [0,1] \to X \times Y$ with

$$\phi(t) = \begin{cases} (\gamma(2t), y_1), & 0 \le t \le 1/2, \\ (x_2, \eta(2(t-1/2))), & 1/2 \le t \le 1. \end{cases}$$

In order to show ϕ is continuous, it is enough to show the coordinate projections are continuous. The first coordinate projection is $\phi_1 : [0,1] \to X$ by

$$\phi_1(t) = \begin{cases} \gamma(2t), & 0 \le t \le 1/2, \\ x_2, & 1/2 \le t \le 1. \end{cases}$$

If U is open in X and $x_2 \notin U$, then $\phi_1^{-1}(U)$ is an open subset of $[0, 1/2]$ that does not contain $1/2$. This is because $\tilde{\gamma}(t) = \gamma(2t)$ is a composition of continuous functions. An open set in $[0, 1/2]$ which does not contain $1/2$ is also an open set in $[0, 1/2)$ and an open set in $[0,1]$.

On the other hand, if $x_2 \in U$, then $\phi_1^{-1}(U)$ is the union of an open set $\tilde{\gamma}^{-1}(U)$ which does contain $1/2$ and the set $[1/2, 1]$. This set is easily seen to be open in $[0,1]$.

We have shown the first coordinate function ϕ_1 is continuous. The second coordinate function $\phi_2 : [0,1] \to Y$ by

$$\phi_2(t) = \begin{cases} y_1, & 0 \le t \le 1/2, \\ \eta(2(t-1/2)), & 1/2 \le t \le 1. \end{cases}$$

is continuous by a very similar argument. It follows that ϕ is continuous and ϕ is a path connecting (x_1, y_1) to (x_2, y_2). Therefore, $X \times Y$ is path connected and, therefore, connected.

For a "cleaner" proof that ϕ is continuous, see Lemma 4.6 on page 69 of Armstrong.