Math 4431, Exam 1: Chapters 1-2 (practice) Name and section: ____________________

1. Let X and Y be topological spaces.

   (a) (10 points) Give a precise definition of continuity for a function $f : X \to Y$.

   (b) (10 points) (pointwise continuity) Show that if $f : X \to Y$ is continuous, then for each $x_0 \in X$ and each open set V in Y with $f(x_0) \in V$, there is some open set U in X with $x_0 \in U$ and $f(U) = \{f(x) : x \in U\} \subset V$.

Solution:

(a) A function $f : X \to Y$ is continuous if $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is open in X whenever V is open in Y.

(b) $f^{-1}(V)$ us such a set.

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2. (2.2.13) A topological space X is called **Hausdorff** if given x and y in X with $x \neq y$, there are disjoint open sets U and V with $x \in U$ and $y \in V$.

   (a) (10 points) Define the term **metric space**.

   (b) (10 points) Show that every metric space is Hausdorff.

Solution:

(a) A **metric space** is a set together with a function $d : X \times X \to [0, \infty)$ satisfying the following for each $x, y, z \in X$

(i) $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$.

(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

(b) Since the metric is positive definite and $x \neq y$, we know $d(x, y) > 0$. Let $r = d(x, y)/2$. Then $B_r(x)$ and $B_r(y)$ are disjoint open sets with $x \in B_r(x)$ and $y \in B_r(y)$. In fact, if $\xi \in B_r(x) \cap B_r(y)$, then

$$d(x, y) \leq d(x, \xi) + d(\xi, y) < 2r = d(x, y).$$

(This is a contradiction.)

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3. (20 points) (2.2.18) If $X = \cup_{j=1}^{\infty} A_j$ and Y are topological spaces and $A_1 \subset \text{int}(A_2) \subset A_2 \subset \text{int}(A_3) \subset A_3 \subset \cdots$, then show that $f : X \to Y$ is continuous if

$$f\big|_{A_j} : A_j \to Y \qquad \text{is continuous for } j = 1, 2, 3, \ldots.$$

Solution: Let V be open in Y and denote the restriction of f to A_j by f_j. Then $f_j^{-1}(V)$ is open in A_j. This means there is a set U_j open in X with $f_j^{-1}(V) = A_j \cap U_j$. Notice that

$$\begin{aligned} f^{-1}(V) &= \cup_{j=1}^{\infty} f^{-1}(V) \cap A_j \\ &= \cup_{j=1}^{\infty} [f_j^{-1}(V) \cap A_j] \\ &= \cup_{j=1}^{\infty} [U_j \cap A_j]. \end{aligned}$$

One appears to be stuck here precisely because we do not know the sets A_j are open. However, because of the nesting, we do know that $X = \cup_{j=1}^{\infty} \text{int}(A_j)$. In order to repeat the basic argument above, we will also need to know

$$f\big|_{\text{int}(A_j)} : \text{int}(A_j) \to Y \qquad \text{is continuous for } j = 1, 2, 3, \ldots.$$

Let's verify this first: If V is open in Y and g_j denotes the restriction of f to $\text{int}(A_j)$, then

$$g_j^{-1}(V) = f_j^{-1}(V) \cap \text{int}(A_j).$$

Since we know f_j is continuous, we know $f_j^{-1}(V)$ is open in A_j. That is, there is some U open in X with $f_j^{-1}(V) = A_j \cap U$. Thus,

$$g_j^{-1}(V) = f_j^{-1}(V) \cap \text{int}(A_j) = U \cap \text{int}(A_j),$$

and this set is open in X. Therefore, we get an even easier proof:

$$f^{-1}(V) = \cup_{j=1}^{\infty} f^{-1}(V) \cap \text{int}(A_j) = \cup_{j=1}^{\infty} [g_j^{-1}(V) \cap \text{int}(A_j)].$$

This is a union of open sets in X and is, therefore, open.

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4. (20 points) Show that given x_0 fixed in a metric space X (with distance function d) the function $f : X \to \mathbb{R}^1$ by $f(x) = d(x, x_0)$ is continuous.

Solution: We can use pointwise continuity here. Let $x_1 \in X$ and let $\epsilon > 0$. Taking $\delta = \epsilon$ and any point x with $d(x, x_1) < \delta = \epsilon$ we can use the triangle inequality

$$d(x, x_0) \leq d(x, x_1) + d(x_1, x_0)$$

to conclude

$$d(x, x_0) - d(x_1, x_0) \leq d(x, x_1) + d(x_1, x_0) - d(x_1, x_0) = d(x, x_1) < \epsilon,$$

and

$$d(x_1, x_0) - d(x, x_0) \geq d(x_1, x_0) - [d(x, x_1) + d(x_1, x_0)] = -d(x, x_1) > -\epsilon.$$

Therefore,

$$|f(x) - f(x_1)| = |d(x, x_0) - d(x_1, x_0)| < \epsilon.$$

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5. (20 points) If A is a (nonempty) closed set in a metric space X and $x \in X \backslash A$, then show $d(x, A) > 0$.

Solution: We know

$$d(x, A) = \inf_{a \in A} d(x, a).$$

Thus, $d(x, A) \geq 0$, and if $d(x, A) = 0$, we have for any $\epsilon > 0$, there is some $a \in A$ with $d(x, a) < \epsilon$. This means $A \cap B_\epsilon(a) \neq \phi$. Therefore, $x \in \text{clus}(A) \subset \overline{A} = A$. This contradicts the fact that $x \notin A$.