Math 4431, Final Exam (practice) Name/Section: ____________________

1. (product space with finitely many factors) Let X_1 and X_2 be topological spaces and for $A_j \subset X_j$, $j = 1, 2$, define

$$A_1 \times A_2 = \{(x_1, x_2) : x_j \in A_j, j = 1, 2\}.$$

Let

$$\mathcal{B} = \{U_1 \times U_2 : U_j \text{ is an open set in } X_j,\ j = 1, 2\}.$$

(a) (5 points) Show that

$$\bigcup_{B \in \mathcal{B}} B = X_1 \times X_2 \quad \text{and} \quad \bigcap_{j=1}^{k} B_j \in \mathcal{B} \quad \text{whenever } B_j \in \mathcal{B},\ j = 1, \ldots, k.$$

(b) (5 points) Show that

$$\mathcal{P} = \left\{ \bigcup_{\alpha \in \Gamma} B_\alpha : B_\alpha \in \mathcal{B} \text{ for } \alpha \text{ in any index set } \Gamma \right\}$$

is a topology on $X_1 \times X_2$. ($\mathcal{P}$ is, of course, called the **product topology**).

(c) (5 points) (Theorem 3.12) Consider $p_j : X_1 \times X_2 \to X_j$ for $j = 1, 2$ by $p_j(x_1, x_2) = x_j$. Show p_1 and p_2 are continuous.

(d) (5 points) Show that if $\mathcal{T}$ is a topology on $X_1 \times X_2$ (not necessarily the product topology $\mathcal{P}$) and p_1 and p_2 are continuous with respect to $\mathcal{T}$, then $\mathcal{P} \subset \mathcal{T}$.

(e) (5 points) Give an example of a topology on $\mathbb{R} \times \mathbb{R}$ with respect to which $p_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is **not** continuous.

(f) (5 points) Give an example of a topology $\mathcal{T}$ which is on $\mathbb{R} \times \mathbb{R}$ which is different from the Euclidean topology but with respect to which $p_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous for $j = 1, 2$.

Solution:

(a) $X_1 \times X_2 \in \mathcal{B}$, and $\cap(U_{1j} \times U_{2j}) = (\cap U_{1j}) \times (\cap U_{2j})$.

(b) $\phi = \phi \times \phi$ and $X_1 \times X_2$ are basic open sets, so $\phi, X_1 \times X_2 \in \mathcal{P}$.

$$\bigcup_\beta \left[\bigcup_\alpha \left(U_{1\alpha}^\beta \times U_{2\alpha}^\beta \right) \right] = \bigcup_{\alpha,\beta} \left(U_{1\alpha}^\beta \times U_{2\alpha}^\beta \right).$$

$$\bigcup_j \left[\bigcup_\alpha (U_{1\alpha}^j \times U_{2\alpha}^j) \right] = \bigcup_\alpha \left[\left(\bigcap_j U_{1\alpha}^j \right) \times \left(\bigcap_j U_{2\alpha}^j \right) \right].$$

(c) $p_1^{-1}(U_1) = U_1 \times X_2$ is open when $U_1 \subset X_1$ is open.

(d) Here we know $U_1 \times X_2, X_1 \times U_2 \in \mathcal{T}$. Therefore,

$$(U_1 \times X_2) \cap (X_1 \times U_2) = U_1 \times U_2 \in \mathcal{T}.$$

Therefore, $\mathcal{B} \subset \mathcal{T}$ and $\mathcal{P} \subset \mathcal{T}$.

(e) We know the topology must be smaller than $\mathcal{P}$. As long as there is some open set $U_1 \neq \phi, X_1$ in X_1, then the topology

$$\{X_1 \times U_2 : U_2 \text{ is open in } X_2\}$$

should be an example. In particular, this should work for $X_1 = X_2 = \mathbb{R}$.

(f) Now, we know the topology should be bigger than $\mathcal{P}$. The discrete topology $2^{X_1 \times X_2}$ will be different from $\mathcal{P}$ as long as X_1 and X_2 do not both have discrete topologies. This, of course, works for $\mathbb{R}^2$.

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2. Let X_1 and X_2 be topological spaces.

   (a) (10 points) (Theorem 3.14) If X_1 and X_2 are Hausdorff, then show $X_1 \times X_2$ is Hausdorff.

   (b) (10 points) (Theorem 3.15) If $X_1 \times X_2$ is compact, then show X_1 and X_2 are compact.

Solution:

(a) Given $(x_1, x_2) \neq (\xi_1, \xi_2)$ in $X_1 \times X_2$, we have either $x_1 \neq \xi_1$ in X_1 or or $x_2 \neq \xi_2$ in X_2. Take the first case. Then there are disjoint open sets U_1 and V_1 in X_1 with $x_1 \in U_1$ and $\xi_1 \in V_1$. The sets $U_1 \times X_2$ and $V_1 \times X_2$ are then disjoint open sets in $X_1 \times X_2$ separating (x_1, x_2) and (ξ_1, ξ_2). The second case is similar.

(b) The projections p_1 and p_2 are continuous and the continuous image of a compact set is compact. Therefore, $X_1 = p_1(X_1 \times X_2)$ is compact. $X_2 = p_2(X_1 \times X_2)$ is compact for the same reason.

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3. (Theorem 3.20) Let us take Armstrong's definition of a connected space:

X is **connected** *if whenever $X = X_1 \cup X_2$ and $X_1, X_2 \neq \phi$, then either*

$$\overline{X}_1 \cap X_2 \neq \phi \quad \text{or} \quad X_1 \cap \overline{X}_2 \neq \phi.$$

(a) (5 points) Show that if $A \subset X$ is connected, then whenever $A \subset A_1 \cup A_2$ and $A \cap A_j \neq \phi$, $j = 1, 2$, then either

$$\overline{A}_1 \cap A_2 \neq \phi \quad \text{or} \quad A_1 \cap \overline{A}_2 \neq \phi.$$

(b) (5 points) Show that if whenever $A \subset A_1 \cup A_2$ and $A \cap A_j \neq \phi$, $j = 1, 2$, then either

$$\overline{A}_1 \cap A_2 \neq \phi \quad \text{or} \quad A_1 \cap \overline{A}_2 \neq \phi,$$

then A is connected.

(c) (5 points) Show that if A is a connected subset of X and $A \subset U_1 \cup U_2$ where U_1 and U_2 are disjoint open sets, then either

$$A \subset U_1 \quad \text{or} \quad A \subset U_2.$$

(d) (5 points) (Corollary 3.24) Show that if A is a connected subspace of X and

$$A \subset S \subset \overline{A},$$

then S is connected.

Solution:

(a) If $A \subset A_1 \cup A_2$, then we know $A = (A_1 \cap A) \cup (A_2 \cap A)$. By the definition of what it means for A to be connected, we have

$$\overline{A \cap A_1} \cap A_2 \neq \phi \quad \text{or} \quad A_1 \cap \overline{A \cap A_2} \neq \phi.$$

In the first case, since

$$\overline{A \cap A_1} \subset \overline{A}_1,$$

we must have $\overline{A}_1 \cap A_2 \neq \phi$. The second case implies $A_1 \cap \overline{A}_2 \neq \phi$.

(b) Again, if $A \subset A_1 \cup A_2$, then $A = (A_1 \cap A) \cup (A_2 \cap A)$, and the assumed conditions are just the definition of what it means for A to be connected (as a space).

(c) If U_1 and U_2 are disjoint open sets, then $\overline{U}_1 \cap U_2 = \phi$. This is because if $x \in U_2$, then U_2 is an open set disjoint from U_1, hence $x \notin \overline{U}_1$. Similarly, $U_1 \cap \overline{U}_2 = \phi$. Now, if we apply part (a) to $A \subset U_1 \cup U_2$, we must have $A \cap U_1 = \phi$ or $A \cap U_2 = \phi$. In the first case, $A \subset U_2$ and in the second case $A \subset U_1$.

The converse of the assertion in part (c), namely:

> *If $S \subset U_1 \cup U_2$ where U_1 and U_2 are disjoint open sets always implies $S \subset U_1$ or $S \subset U_2$, then S is connected.*

is **false**.

To see this, consider $\mathcal{T} = \{\phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. This is a topology on $X = \{a, b, c\}$. The set $S = \{a, c\}$ satisfies the condition of the converse of (c). This is because the only pair of disjoint open sets whose union contains S is the pair consisting of ϕ and $X = \{a, b, c\}$. On the other hand, S is not connected because $S \cap \{a, b\} = \{a\}$ and $S \cap \{b, c\} = \{c\}$ are open **relative to** S.

(d) Assume S is not connected. Then there are sets S_1 and S_2 with $S \subset S_1 \cup S_2$,

$$S \cap S_j \neq \phi, \ j = 1, 2, \qquad (1)$$

and

$$\overline{S}_1 \cap S_2 = \phi = S_1 \cap \overline{S}_2.$$

Since $A \subset S \subset S_1 \cup S_2$ and A **is** connected, we must have $A \subset S_1$ or $A \subset S_2$. In the first case, $\overline{A} \subset \overline{S}_1$ and, consequently, $\overline{A} \cap S_2 = \phi$. But since $S \subset \overline{A}$, this means $S \cap S_2 = \phi$ which contradicts (1). In the second case, we obtain a similar contradiction since then $S \subset \overline{A} \subset \overline{S}_2$, and it follows that $S \cap S_1 = \phi$.

Since we have contradictions in all cases, our assumption that S is not connected must be bogus. S must be connected.

4. A topological space X is **locally connected** if for each $x \in X$ and each open set U with $x \in U$, there is some open set U_0 and some connected set C with

$$x \in U_0 \subset C \subset U.$$

(a) (10 points) Show that the homeomorphic image of a locally connected space is locally connected.

(b) (10 points) Show that if X is locally connected, then for each $x \in X$ and each open set U with $x \in U$, there is an open connected set U_0 with

$$x \in U_0 \subset U.$$

Solution:

(a) If $h : X \to Y$ and X is locally connected, then given any point $y = h(x) \in Y$ and an open set V with $y \in V$, we have a point $x \in X$, and we want to apply the definition of local connectedness of X at x. We can take the open set $U = h^{-1}(V)$, and we get an open set U_0 and a connected set C with $x \subset U_0 \subset C \subset U$. Then we have an open set $h(U_0)$ and a connected set $h(C)$ with

$$y \in h(U_0) \subset h(C) \subset V.$$

This means $h(X)$ is locally connected.

(b) We cannot take U_0 directly from the definition, because U_0 may not be connected. What we can do is take a set U_1 from the definition, and we'll take the connected set C too, with

$$x \in U_1 \subset C \subset U.$$

Now, we can take U_0 to be the component of U_1 containing x. Let us denote this set $U_0 = \text{comp}_x(U_1)$. We need to show U_0 is open. The component $\text{comp}_x(U_1)$ is the union of all connected subsets of U_1 containing x, and it follows from this that U_0 is a connected subset of U_1. (At this point, you might be tempted to take U_0 as the union of all **open** connected subsets of U_1, but you wouldn't yet know there are any such sets, so you'd still be stuck.) The good news is that all we have to show is that U_0 is open.

Take a point $\xi \in U_0$. We know then, since X is locally connected, that there is an open set U_ξ and a connected set C_ξ with

$$\xi \in U_\xi \subset C_\xi \subset U_1.$$

Since C_ξ is connected with $\xi \in C_\xi \subset U_1$, and $\text{comp}_\xi(U_1)$ is the union of all such sets, we know

$$\xi \in C_\xi \subset \text{comp}_\xi(U_1).$$

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On the other hand, $\xi \in \text{comp}_x(U_1)$ which is also a connected subset of U_1 containing ξ. Therefore, $\text{comp}_x(U_1) \subset \text{comp}_\xi(U_1)$. In particular, $x \in \text{comp}_\xi(U_1)$. It follows in the same way that $\text{comp}_\xi(U_1) \subset \text{comp}_x(U_1)$. In particular,

$$\xi \in U_\xi \subset \text{comp}_x(U_1) = U_0. \tag{2}$$

The existence of such an open set U_ξ for every ξ shows U_0 is open (and we're done).

The little argument above, starting with "On the other hand" and continuing up to (2) essentially shows that if $\xi \in \text{comp}_x(U_1)$, then $\text{comp}_\xi(U_1) = \text{comp}_x(U_1)$, that is, components are disjoint connected sets partitioning whatever set you take the components in (in this case U_1). This fact could also be quoted in this problem, if you remember it.

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5. Let $X = (0, 1]$ and consider $\phi : X \to \mathbb{R}^2$ by

$$\phi(t) = \begin{cases} (t, \sin(1/t)), & 0 < t \leq 2/\pi \\ (6/\pi - 2t, 1), & 2/\pi \leq t \leq 3/\pi \\ (0, 7 - 2\pi t), & 3/\pi \leq t \leq 1. \end{cases}$$

Let $Y = \phi(X)$.

(a) (5 points) Show X is locally path connected.

(b) (5 points) Show ϕ is continuous so that Y is the continuous image of a locally path connected space.

(c) (5 points) Show Y is **not** locally path connected.

(d) (5 points) Show the homeomorphic image of a locally path connected space is locally path connected.

Solution:

(a) Recall that a space X is **locally path connected** if for each $x \in X$ and each open set U with $x \in U$, there is an open set U_0 and a path connected set C with $x \in U_0 \subset C \subset U$.

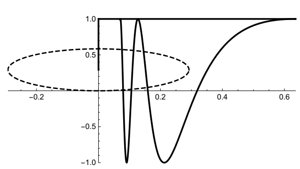
If $0 < a < b \leq 1$, then $\gamma(t) = (1-t)a + tb$ is a path from a to b. Or we could just remember that intervals are path connected. In any case, the same is true for any open interval, so given an open set U and a point $x \in U$, there is an open interval U_0 with $X \in U_0 \subset U$. The interval U_0 is open and path connected.

(b) Taking $t = 2/\pi$ in the first case of ϕ, we get $(2/\pi, 1)$. The same value of t in the second case gives $(2/\pi, 1)$.

Taking $t = 3/\pi$ in the second case gives $(0, 1)$. The same value of t in the third case gives $(0, 1)$.

Since these values agree, ϕ is well-defined. Furthermore, ϕ is continuous by the gluing lemma.

(c) The space Y looks like this:



If we take an open ball V centered as pictured at the endpoint $\phi(1) = (0, 7 - 2\pi) = p$ and having small radius, then $V \cap \phi(X)$ contains infinitely many components. If p is in any open set V_0 with $V_0 \subset V$, then there is no connected set C with $V_0 \subset C \subset V$. This is because infinitely many of the components of V must also intersect V_0, but any connected set in V must be a subset of only

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one component. Thus, $\phi(X)$ is not even locally connected. (Since local path connectedness implies local connectedness, this means $\phi(X)$ is not locally path connected.)

(d) This is quite similar to part (a) of the previous problem.

If $h : X \to Y$ and X is locally path connected, then given any point $y = h(x) \in Y$ and an open set V with $y \in V$, we have a point $x \in X$, and we want to apply the definition of local path connectedness of X at x. We can take the open set $U = h^{-1}(V)$, and we get an open set U_0 and a connected set C with $x \subset U_0 \subset C \subset U$. Then we have an open set $h(U_0)$ and a path connected set $h(C)$ with

$$y \in h(U_0) \subset h(C) \subset V.$$

This means $h(X)$ is locally path connected.

We used here that the continuous image of a path connected set is path connected, which is true.

As this argument was pretty easy/straightforward, and the continuous images of path connected spaces are path connected, it is interesting to see where it breaks down for $\phi(X)$. A quick look at the argument shows that the only questionable point is the assertion that the forward image $h(U_0)$ is an **open set**. This must fail for $\phi(U_0)$. Let's see: V was the ball shown in the drawing. The inverse image of V is the union of some interval $(b, 1]$ with an infinite collection of open intervals to the left of $(b, 1]$. This is $U = \phi^{-1}(V)$, and indeed, this is an open set. Also, $1 \in U$ with $\phi(1) = (0, 7-2\pi) = p$ the center of V. We can apply local connectedness (or local path connectedness) in $X = (0, 1]$ and take an open interval $U_0 = (b', 1] \subset (b, 1]$. In fact, $U_0 = (b, 1]$ will be fine. Then we see the problem, the forward image $\phi(b, 1]$ is definitely not open in Y. This is a vertical segment

$$\{(0, t) : a < t \leq 7 - 2\pi\}.$$

The point p, in particular, is in this vertical segment, and any open set about p, as mentioned above, contains many components of $\phi(X)$ outside of the vertical segment.

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6. Let $q : \mathbb{R} \to \{-1, 0, 1\} = Y$ by

$$q(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0. \end{cases}$$

Recall that the quotient topology on Y is defined by

$$\mathcal{Q}_Y = \{V \subset Y : q^{-1}(V) \text{ is open in } \mathbb{R}\}.$$

(a) (10 points) What is $\mathcal{Q}_Y$?

(b) (10 points) (True or False) If $q : X \to Y$ is an identification map and X is Hausdorff, then Y is Hausdorff.

Solution:

(a)

$$\mathcal{Q}_Y = \{\phi, \{-1\}, \{-1, 1\}, \{1\}, \{-1, 0, 1\}\}.$$

(There are three singleton sets, and $\{0\}$ has inverse image $\{0\}$ which is not open in $\mathbb{R}$. There are three doubleton sets. If a doubleton has 0 in it, then the inverse image is a closed interval $[0, \infty)$ or $(-\infty, 0]$. In neither case is the inverse image open.)

(b) This map q is continuous, essentially by definition, and $\mathbb{R}$ is certainly Hausdorff. However, there is no open set separating 0 and 1 in Y. So the assertion is **False**.

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7. Let $\mathbb{Z} \times \mathbb{Z} = \{(m, n) : m, n \text{ are integers}\} \subset \mathbb{R}^2$.

   (a) (5 points) Show $G = \mathbb{Z} \times \mathbb{Z}$ is a group (under addition).

   (b) (5 points) Show G acts on $\mathbb{R}^2$ by $(n, m, x, y) \mapsto (x + n, y + m)$.

   (c) (5 points) Identify the quotient space $\mathbb{R}^2/G$.

   (d) (5 points) Consider $A = \mathbb{Z} \times \mathbb{Z} \subset \mathbb{R}^2$. Show that the quotient spaces $\mathbb{R}^2/G$ and $\mathbb{R}^2/A$ are **not** homeomorphic.

   This problem needs an adjustment/correction. The first three parts are okay, but the assertion of part (d) is not so obvious. Let's drop a dimension, and make our group $\mathbb{Z}$. Of course part (a), then becomes trivial. Part (b) has the same solution as below, just given componentwise, so it is a bit easier—just checking the definition. The space in part (c) changes as described in the solution of part (d) below.

Solution:

(a) The operation is $(m, n) + (\mu, \nu) = (m + \mu, n + \nu)$. This is clearly well-defined and associative. The identity element is $(0, 0)$. The inverse of (m, n) is $(-m, -n)$.

(b) The suggested function is clearly a well-defined function from $\mathbb{Z}^2 \times \mathbb{R}^2$ to $\mathbb{R}^2$. Also,

$$(m + \mu, n + \nu)(x, y) = (m + \mu + x, n + \nu + y) = (m, n)(\mu + x, \nu + y).$$

This is the required associative property. Finally,

$$(0, 0)(x, y) = (x, y),$$

so the identity acts as it should, and we have a group action.

For the definition, see Homework Assignment 10.

(c) $\mathbb{R}^2/G$ is the torus $\mathbb{S}^1 \times \mathbb{S}^1$.

(d) In view of the correction, The first space we want to consider is $\mathbb{R}/\mathbb{Z} = \mathbb{S}^1$, the circle. The other space $Y = \mathbb{R}/\mathbb{Z}$ with $\mathbb{Z}$ considered as a subset, on the other hand, is a countably infinite collection of circles joined at one point. When you remove a point from the circle, what you have left is connected. When you remove the common point from Y, the image of each interval $(j, j+1)$ for $j \in \mathbb{Z}$ is a distinct connected component, so these spaces can't be homeomorphic.

To be a bit more precise, $Y = \mathbb{R}/\mathbb{Z}$ is a partition of $\mathbb{R}$ consisting of partition sets $\mathbb{Z}$ and $\{x\}$ where $x \notin \mathbb{Z}$. (These are the points in $Y = \mathbb{R}/\mathbb{Z}$.) Therefore, if we assume $h : \mathbb{S}^1 \to Y = \mathbb{R}/\mathbb{Z}$ is a homeomorphism. Then $\mathbb{S}^1 \backslash \{h^{-1}(\mathbb{Z})\}$ is connected, but $(\mathbb{R}/\mathbb{Z}) \backslash \mathbb{Z}$ has countably many components $(j, j + 1)$ for $j \in \mathbb{Z}$.

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8. Consider the (universal) covering map $\phi : \mathbb{R}^2 \to \mathbb{T}^2$ by

$$\phi(x, y) = \left(1 + \frac{\cos y}{2}\right)(\cos x, \sin x, 0) + \frac{\sin y}{2}(0, 0, 1).$$

Also, consider the loops $\gamma[0, 1] \to \mathbb{T}^2$ and $\eta : [0, 1] \to \mathbb{T}^2$ by $\gamma(t) = 3(\cos 2\pi t, \sin 2\pi t, 0)/2$ and

$$\eta(t) = \left(1 + \frac{\cos 2\pi t}{2}\right)(1, 0, 0) + \frac{\sin 2\pi t}{2}(0, 0, 1)$$

respectively both of which start and end at $p = (3/2, 0, 0)$.

(a) (5 points) Draw the (image sets of the) loops γ and η on $\mathbb{T}^2$.

(b) (5 points) Recall the **concatenation** $\eta \triangleleft \gamma : [0, 1] \to \mathbb{T}^2$ is a loop given by

$$\eta \triangleleft \gamma(t) = \begin{cases} \gamma(2t), & 0 \le t \le 1/2 \\ \eta(2t - 1), & 1/2 \le t \le 1. \end{cases}$$

Find explicit formulas for the **liftings** of γ, η, $\eta \triangleleft \gamma$, and $(-\gamma) \triangleleft \eta \triangleleft \gamma$ starting at $(0, 0) \in \mathbb{R}^2$, and draw these paths.

(c) (5 points) Find a fixed endpoint homotopy of the lifting

$$\widehat{(-\gamma) \triangleleft \eta \triangleleft \gamma}$$

to a path along a straight line. To which of the four liftings of part (c) above is this straight line path homotopic?

Solution: