**Math 4431, Final Exam** (practice) Name/Section: ________________

1. (20 points) (Topologist's Isoperimetric Inequality) The isoperimetric inequality for subsets of $\mathbb{R}^n$ relates the $n$-dimensional measure of a measureable set $A \subset \mathbb{R}^n$ to the $n-1$-dimensional measure of it boundary:

$$\frac{[\mathcal{H}^{n-1}(\partial A)]^n}{[m_n(A)]^{n-1}} \geq \frac{[\mathcal{H}^{n-1}(\partial B_1)]^n}{[m_n(B_1)]^{n-1}}$$

where $m_n$ is $n$-dimensional Lebesgue measure on $\mathbb{R}^n$, $\mathcal{H}^{n-1}$ is $n-1$-dimensional Hausdorff measure on $\mathbb{R}^n$, and $B_1$ is any ball in $\mathbb{R}^n$ of unit radius. For example, if $n = 2$, then this says the length of the boundary squared divided by the area of a set $A$ is at least $4\pi$.

Prove the following inequality for any subset $A$ of a topological space $X$:

$$\frac{[\nu(\partial A)]^q}{[\mu(A)]^p} \geq \frac{[\nu(\partial \overline{A})]^q}{[\mu(\overline{A})]^p} \tag{1}$$

where $\overline{A}$ is the closure of $A$, $\mu : 2^X \to [0, \infty]$ and $\nu : 2^X \to [0, \infty]$ are any nonnegative **monotone set functions**, and $p$ and $q$ are any nonnegative real numbers. (A **monotone set function** $\mu$ is one for which $\mu(A) \leq \mu(B)$ whenever $A \subset B$.)

**Solution:** We first claim that

$$\partial \overline{A} \subset \partial A. \tag{2}$$

To see this, let $x \in \partial \overline{A}$. Then for every open set $U$ with $x \in U$ we have some $\xi \in U \cap \overline{A}$ and some $\eta \in U \cap (\overline{A})^c$. Since $A \subset \overline{A}$, we know $\eta \in U \cap A^c$. Also, since $U$ is an open set containing $\xi$ and $\xi \in \overline{A}$, there is some $a \in A \cap U$. This means $x \in \partial A$, and we have established (2).

Since $\nu$ is monotone we get

$$\nu(\partial A) \geq \nu(\partial \overline{A}). \tag{3}$$

On the other hand, it is clear that $A \subset \overline{A}$, so by the monotonicity of $\mu$ we have

$$\mu(A) \leq \mu(\overline{A}). \tag{4}$$

Combining (3) and (4) with some arithmetic of the extended real numbers, we get (1).

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2. To the **left** of each term write the number of the appropriate definition/explanation.

   (a) (2 points) connected

   (b) (2 points) compact

   (c) (2 points) locally connected

   (d) (2 points) locally compact

   (e) (2 points) path connected

   (f) (2 points) locally path connected

   (g) (2 points) product space

   (h) (2 points) closure

   (i) (2 points) open

   (j) (2 points) closed

   1. an element of the topology
   2. the set of all functions
   $$f : \Gamma \to \bigcup_{\alpha \in \Gamma} X_\alpha$$
   where $\{X_\alpha\}$ is a collection of topological spaces with index set $\Gamma$ and $f(\alpha) \in X_\alpha$
   3. for a set $A$,
   $$\bigcap_{A \cap U = \phi,\ U \text{ open}} U^c$$
   4. for each point $x$, there is an open set $U$ and a compact set $K$ such that $x \in U \subset K$
   5. whenever $X = A \cup B$, then either $\overline{A} \cap B \neq \phi$ or $A \cap \overline{B} \neq \phi$
   6. for each pair $(a, b)$, there is a continuous function $f$ defined on an interval $[0, 1]$ such that $f(0) = a$ and $f(1) = b$
   7. when the set $\cap A_\alpha$ is an intersection of closed sets and $\cap A_\alpha \cap A = \phi$, then one can find a finite collection $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ for which
   $$\bigcap_{j=1}^{k} A_{\alpha_j} \cap A = \phi$$
   8. for each $x$ and each open set $U$ with $x \in U$, there is a path connected set $C$ and an open set $W$ with $x \in W \subset C \subset U$
   9. $A^c$ is an open neighborhood
   10. whenever $U$ is open and $x \in U$, there is an open set $W$ with $x \in W \subset U$ and every nonempty proper open subset of $W$ has nonopen complement

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**Solution:**

(a) 5

(b) 7

(c) 10

(d) 4

(e) 6

(f) 8

(g) 2

(h) 3

(i) 1

(j) 9

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3. To the **left** of each term write the number of the appropriate definition/explanation.

   (a) (2 points) second countable

   (b) (2 points) Hausdorff

   (c) (2 points) loop

   (d) (2 points) homotopy

   (e) (2 points) fundamental group

   (f) (2 points) deformation retraction

   (g) (2 points) group

   (h) (2 points) continuity

   (i) (2 points) identification map

   (j) (2 points) topologist's sine curve

   1. a homotopy of a set into itself
   2. associative loop concatenation in a path connected space
   3. a continuous function such that when the inverse image of a set is open, then the set is also open
   4. having a basis of open sets $U_1, U_2, U_3, \ldots$
   5. a function $f$ defined on an interval $[0,1]$ and satisfying $f(0) = f(1)$
   6. example of a connected space whose closure is not path connected
   7. a continuous function on the cross product of a space with the interval $[0,1]$
   8. the inverse image of an open set is open
   9. having an associative operation on a set that contains an identity element and inverses
   10. being able to separate points by open sets

**Solution:**

(a) 4

(b) 10

(c) 5

(d) 7

(e) 2

(f) 1

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(g) 9

(h) 8

(i) 3

(j) 6

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4. Let $X = \mathbb{R}^2 \backslash \{(0,0)\}$.

   (a) (10 points) Identify the fundamental group of $X$ and give explicit representatives (loops) for each element of the fundamental group.

   (b) (10 points) Take one of your loop representatives $\gamma$ and a representative $\eta$ of $\langle\gamma\rangle^{-1}$, and give an explicit homotopy of the **concatenation** $\eta \triangleleft \gamma$ to the identity. Here, the **concatenation** is defined by

$$\eta \triangleleft \gamma(t) = \begin{cases} \gamma(2t), & 0 \leq t \leq 1/2 \\ \eta(2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

**Solution:**

(a) The fundamental group is $\mathbb{Z}$. Let $\gamma : [0,1] \to \mathbb{R}^2 \backslash \{(0,0)\}$ by $\gamma(t) = (\cos 2\pi t, \sin 2\pi t)$. Denote by $\gamma^j$ the loop $\gamma$ concatenated with itself $j$ times for $j = 2, 3, 4, \ldots$. The constant loop $\mathrm{id}(t) \equiv (1,0)$ represents the identity in the fundamental group. The remaining elements are the inverses of $\gamma^j$:

$$\gamma^{-1} = -\gamma : [0,1] \to \mathbb{R}^2 \backslash \{(0,0)\} \quad \text{by} \quad \gamma^{-1}(t) = (\cos 2\pi t, -\sin 2\pi t),$$

and let $\gamma^{-j}$ be $\gamma^{-1}$ concatenated with itself $j$ times for $j = 2, 3, 4, \ldots$. The mapping $\langle\gamma^j\rangle \mapsto j$ is an isomorphism of $\pi_1(X) \to \mathbb{Z}$.

(b) Take $\gamma(t) = (\cos 2\pi t, \sin 2\pi t)$ and $\eta(t) = \gamma^{-1}(t) = \gamma(1-t) = (\cos 2\pi t, -\sin 2\pi t)$. Define

$$H(t,s) = \begin{cases} \gamma(2(1-s)t), & 0 \leq t \leq 1/2 \\ \eta((1-s)(2t-1)+s), & 1/2 \leq t \leq 1. \end{cases}$$

Notice first that the values assigned to $H$ for $t = 1/2$ are $\gamma(1-s) = \eta(s)$. Thus, $H$ is well-defined and continuous by the gluing lemma. Also, $H(0,s) = \gamma(0) \equiv (1,0)$ and $H(1,s) = \eta(1) \equiv (1,0)$. Finally, $H(t,0) \equiv \eta \triangleleft \gamma$ and $H(t,1) \equiv (1,0)$. Thus, $H$ is a homotopy of $\eta \triangleleft \gamma$ to the identity (loop).

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5. (20 points) Recall that $X$ is said to be **locally compact** if for each point $x \in X$, there is an open set $U$ and a compact set $K$ such that $x \in U \subset K$. Assume $X$ is locally compact and Hausdorff. Show that given any point $x \in X$ and any open set $U$ with $x \in U$, there is a compact set $K$ and an open set $W$ with $x \in W \subset K \subset U$.

**Solution:** Because $X$ is locally compact, we can start with an open set $U_0$ and a compact set $K_0$ such that $x \in U_0 \subset K_0$. We can, of course, take $W_0 = U \cap U_0$ which is an open set with $x \in W_0 \subset K_0$. We have no reason to believe, however, that $K_0 \subset U$.

Because $X$ is Hausdorff, we do know $K_0$ is closed. And furthermore, if we had another closed subset $C$ of $U$, then $K_0 \cap C$ would be a compact subset of $U$. (Closed subsets of compact sets are always compact.)

Consider $\{x\}$ and $U^c$. These are both closed sets. In particular, $K_1 = K_0 \cap U^c$ is compact. Of course, this intersection $K_1$ might be empty, but if it is, that means $K_0 \subset U$ and we're done because we can just take $W = U_0 \subset K = K_0 \subset U$.

Otherwise, for each point $\xi$ in $K_1$, there are disjoint open sets $U_\xi$ and $V_\xi$ with $x \in U_\xi$ and $\xi \in V_\xi$. Taking a finite subcover of $K_1$ consisting of a finite collection of the $V_\xi$, we get disjoint open sets $U_1$ (the intersection of the corresponding $U_\xi$) and $V$ (the union of the finitely many $V_\xi$) with $x \in U_1$ and $K_1 \subset V$.

This last paragraph is just a proof that you can separate a point from a compact set in a Hausdorff space. That fact can be quoted if you remember it.

In any case, $K = K_0 \cap V^c$ is our compact set, and $W = U_0 \cap U_1$ is our open set. There are, perhaps, some things to check. First of all $K$ is compact because, as mentioned, $K_0$ and $V^c$ are closed making $K$ a closed subset of the compact set $K_0$. (A closed subset of a compact set is always compact.) The set $W$ is also open and nonempty because $x \in W$. Also, since $U_0 \subset K_0$ and $U_1 \subset V^c$, we know $W \subset K$. It remains to show $K \subset U$. But remember that if $\xi \in K$, then $\xi \in V^c$ and this means, $\xi \notin K_1 = K_0 \cap U^c$. Since we know $\xi \in K_0$, it must be that $\xi \notin U^c$, i.e., $\xi \in U$. This completes the solution.