Complex Analysis:
An Elementary Introduction

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Preface to the Student

These notes from Spring semester 2023 go along with the book *Complex Variables and Applications* by Brown and Churchill (known as "BC" in these notes). I will follow the book pretty closely, but I will often phrase the material in a rather different way. This is intentional. My suggestion is that you read the notes and the book; they are pretty much in some kind of one-to-one correspondence. Work lots of the exercises in these notes and all of the problems in the accompanying assignments.

If you're planning to take a course in complex analysis, then you should probably already be familiar with the appearance of complex numbers in the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1}$$

for the solutions of the quadratic polynomial equation $ax^2 + bx + c = 0$ when

$$b^2 - 4ac < 0.$$

Exercise 1 Derive the quadratic formula (1) by completing the square in $ax^2 + bx + c = 0$.

In this connection, you should probably have at least a little experience simplifying algebraic expressions involving complex numbers. For example, you should not be surprised to know

$$i^2 = -1, \qquad \frac{1}{i} = -i, \qquad \text{and} \qquad (a + bi)(a - bi) = a^2 + b^2 \in \mathbb{R},$$

though you can look forward to becoming much more, shall we say, practiced in making such algebraic manipulations.

You may also be familiar with the important fact from algebra that the set of all complex numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

is a "system of numbers" in which every polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with coefficients $a_0, a_1, \ldots, a_n \in \mathbb{C}$ has a root, i.e., solution, in $\mathbb{C}$. We should be able to go through and understand a proof of this theorem (called the fundamental theorem of algebra) which is presented in BC near the end of Chapter 4.

You probably have run across Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

and may even be able to make sense of a series expansion like

$$e^{a+bi} = \sum_{n=0}^{\infty} \frac{1}{n!} a^n \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} b^{2k} + i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} b^{2k+1} \right).$$

We will discuss many properties and details concerning the extension of the exponential function (and other special functions) to $\mathbb{C}$. It is assumed you are not familiar with this material, and it is one of the main things you should expect yourself to learn during the course.

Most importantly, we will go a bit beyond complex arithmetic to talk about the main subject of complex analysis, which is the **complex function** $f : \mathbb{C} \to \mathbb{C}$, and especially the class of these functions which are **complex differentiable**. I hope you are in a position to appreciate such a discussion. You may have had some opportunity to appreciate similar questions in calculus when you discussed the family of real valued functions $f : (a, b) \to \mathbb{R}$ defined on an open interval (a, b) which are differentiable, twice differentiable, and (eventually) infinitely differentiable, and those represented by a power series, i.e., the real analytic functions, though usually the emphasis in elementary calculus and even some advanced calculus is elsewhere. In reference to calculus, it may also be useful to point out that one object of study in calculus, say third semester calculus, might be the real valued function $f : U \to \mathbb{R}$ where U is some open set in the real plane $\mathbb{R}^2$, like for example the square domain $U = \{(x, y) \in \mathbb{R}^2 : |x| < 1 \text{ and } |y| < 1\}$ or the disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}$. You might have learned to compute the partial derivatives of such a function in order to maximize, minimize, or otherwise understand such a function. These kinds of considerations are not unrelated to the study of complex functions, and we will use some results from multivariable calculus, though we will think about them again carefully, i.e., review them, when the time comes.

Another opportunity for considering functions in some sense like we will attempt to consider them in complex analysis may be found in linear algebra when one considers, for example, linear mappings $L : \mathbb{R}^2 \to \mathbb{R}^2$ of the real plane to itself. Among these mappings are various rotations, scalings, and reflections with which you may be familiar. The point is, first of all, that there are geometric aspects involved with such functions that are very similar to the geometric considerations we will encounter for complex functions in complex analysis, and more generally, you probably have some experience considering questions about functions in general, though presumably not when the domain and codomain of the functions under consideration are subsets of $\mathbb{C}$. In that sense you should be able to look forward to learning (in this course) something about a whole new world of functions with their own particular, and often beautiful, characteristics.

Exercise 2 Identify the functions represented by these power series:

$$u(x) = \sum_{\ell=0}^{\infty} \left(\sum_{m=0}^{M(\ell)} \frac{(-1)^m}{(\ell - 2m)!(2m)!} \right) x^\ell$$

and

$$v(x) = \sum_{\ell=0}^{\infty} \left(\sum_{m=0}^{M(\ell-1)} \frac{(-1)^m}{(\ell - 2m - 1)!(2m+1)!} \right) x^\ell$$

where

$$M(k) = \min \left\{ n \in \mathbb{N} : n \geq \frac{k-1}{2} \right\} \qquad \text{and} \qquad \mathbb{N} = \{1, 2, 3, \ldots\}.$$

Preface

These are notes for MATH 4320 (Undergraduate Complex Analysis) offered in the Spring semester of 2023 at Georgia Institute of Technology. The text for the course was *Complex Variables and Applications* by Brown and Churchill (ninth edition 2014).

Chapter 1

Complex Numbers

1.1 Sums and Products and Basic Algebraic Properties

The **set of complex numbers** is perhaps most simply introduced as

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}. \tag{1.1}$$

The **operations** of addition and multiplication of complex numbers are defined by

$$(a + bi) + (c + di) = a + c + (b + d)i$$

and

$$(a + bi)(c + di) = ac - bd + (ad + bc)i.$$

It is natural to compare this set of complex numbers with two operations to two other familiar sets. These are the set of real numbers $\mathbb{R}$ (which play a role already in the "construction" of the complex numbers above) and the set of ordered pairs of real numbers

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}. \tag{1.2}$$

The Real Numbers

Brown and Churchill[1] who wrote a textbook (your textbook) on complex analysis assume "the various corresponding properties of the real numbers to be known." Let's review some of the algebraic properties of the real numbers:

[1] *Complex Variables and Applications* (ninth edition 2014, first edition 1948) henceforth referred to in these notes as "BC."

1. addition is **associative**:

$$(a+b)+c = a+(b+c).$$

Exercise 1.1 Show addition in $\mathbb{C}$ is associative.

Solution: Let $a+bi$, $c+di$, and $x+iy$ denote three complex numbers with $a,b,c,d,x,y \in \mathbb{R}$.

$$\begin{aligned}(a+bi+c+di)+x+iy &= [a+c+(b+d)i] \;+\; x+iy \\ &= (a+c)+x+[(b+d)+y]i \\ &= a+(c+x)+[b+(d+y)]i \\ &= a+bi+[c+x+(d+y)i] \\ &= a+bi+(c+di+x+iy).\end{aligned}$$

Note that the associativity of addition in $\mathbb{C}$ follows directly from the associativity of addition in $\mathbb{R}$.

2. addition (in $\mathbb{R}$) is **commutative**:

$$a+b = b+a \qquad \text{for} \qquad a,b \in \mathbb{R}.$$

Exercise 1.2 Show addition in $\mathbb{C}$ is commutative.

3. There exists an **additive identity** in $\mathbb{R}$: There is an element $z \in \mathbb{R}$ such that

$$a+z = z+a = a \qquad \text{for all } a \in \mathbb{R}.$$

We call the additive identity in $\mathbb{R}$ "zero" and denote the additive identity in $\mathbb{R}$ by 0.

4. There exist **additive inverses**: For each $a \in \mathbb{R}$, there exists some $b \in \mathbb{R}$ with

$$a+b = b+a = 0.$$

We denote the element b by $-a$.

Any set G with an operation $* : G \times G \to G$ by

$$*(a,b) = a*b$$

satisfying properties 1-4 (with $*$ in place of $+$) is called[2] a **commutative group**. The set $\mathbb{R}$ is a commutative group under addition.

[2]If property 2, the commutativity property, is omitted, then the set G is said to be an **algebraic group** or just a **group** for short.

Exercise 1.3 Rephrase conditions 3 and 4 as they would apply to a general algebraic group G with operation $*$ and show $\mathbb{C}$ is a commutative group under addition.

Exercise 1.4 Show the identity element in any group is unique.

Exercise 1.5 Show the inverse elements in any group are unique.

Solution: If G is a group with identity element z and $a \in G$ with

$$a * b = b * a = z \qquad \text{and} \qquad a * c = c * a = z \tag{1.3}$$

for some elements $b, c \in G$, then

$$c = z * c = (b * a) * c = b * (a * c) = b * z = b. \tag{1.4}$$

Notice this solution does not use all the information given in (1.3). It only uses the existence of an identity element z, associativity, and the existence of elements b and c with $b * a = z$ and $a * c = z$. With this in mind, let me throw in some side comments about algebra.

> In some contexts (that is algebraic contexts) one might consider various sets with less structure than that of a group. For example a **semigroup** is a set S with an associative operation $* : (a, b) \mapsto a * b$. In a semigroup, one may consider the notion of a **left identity** and/or a **right identity**. A **left identity** is an element $z \in S$ such that $z * a = a$ for all $a \in S$. Similarly, a **right identity** is an element $z \in S$ such that $a * z = a$ for all $a \in S$. Notice our definition of an **identity element** is an element which is both a left and right identity.
>
> A **monoid** M is a semigroup with an identity element. In a monoid, one may consider the notion of a **left inverse** and/or a **right inverse**. An element $a \in M$ has a **left inverse** $b \in M$ if $b * a = z$. An element $a \in M$ has a **right inverse** c if $a * c = z$. Now, let's say an element a in a monoid M has a left inverse b and a right inverse c. Then (1.4) implies the left inverse and the right inverse must be the same element. Put another way, given a left inverse b for an element $a \in M$, the existence of a right inverse for that element implies b is a right inverse as well. This is all valid without commutativity. It is crucial to note, however, that just having a left inverse is not enough: If M is the collection of all functions

$f : \mathbb{N} \to \mathbb{N}$ where $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the natural numbers and the monoid operation is function composition. The identity element is just the identity function $z(j) \equiv j$ for $j \in \mathbb{N}$. The function $p(j) = j + 1$ has a left inverse given by

$$q(j) = \begin{cases} j - 1, & j \neq 1 \\ 1, & j = 1. \end{cases}$$

But p has no right inverse.

A **group** may be defined as a monoid in which every element has both a left and a right inverse.

5. Multiplication in $\mathbb{R}$ is associative and commutative.

6. There exists a multiplicative identity. We call this element $1 \in \mathbb{R}$.

   Exercise 1.6 Show multiplication in $\mathbb{C}$ is associative and commutative.

   Exercise 1.7 Show there exists (and identify) a multiplicative identity in $\mathbb{C}$, and show this multiplicative identity is unique.

   Exercise 1.8 Use Exercise 1.4 to show the multiplicative identity in $\mathbb{C}$ is unique.

7. There exist (some) **multiplicative inverses** in $\mathbb{R}$: For each $a \in \mathbb{R}\backslash\{0\}$, there exists an element $b \in \mathbb{R}$ for which

$$ab = ba = 1.$$

   The multiplicative inverse b of $a \in \mathbb{R}\backslash\{0\}$ is written as a^{-1} or $1/a$.

   Exercise 1.9 Show that given $a + bi \in \mathbb{C}\backslash\{0\}$ there exists a multiplicative inverse in $c + di = (a + bi)^{-1} \in \mathbb{C}\backslash\{0\}$.

   Exercise 1.10 Show the multiplicative inverses in $\mathbb{C}\backslash\{0\}$ are unique.

   The final algebraic property we will mention is the following:

8. Multiplication is **distributive** across addition in $\mathbb{R}$:

$$a(b + c) = ab + ac \qquad \text{for all } a, b, c \in \mathbb{R}.$$

Exercise 1.11 Show multiplication is distributive across addition in $\mathbb{C}$:

$$w(z_1 + z_2) = wz_1 + wz_2 \qquad \text{for all } z_1, z_2, w \in \mathbb{C}.$$

Any set F with two binary operations (of addition $(a, b) \mapsto a + b$ and multplication $(a, b) \mapsto ab$) to which the properties listed above for $\mathbb{R}$, and shown to hold in the exercises for $\mathbb{C}$, is called a **field**. This concludes our initial algebraic comparison of $\mathbb{C}$ with $\mathbb{R}$: Both $\mathbb{C}$ and $\mathbb{R}$ are fields.

Exercise 1.12 Show $\mathbb{Z}_3 = \{0, 1, 2\}$ with a commutative addition satisfying[3]

$$\begin{array}{rcll} 0 + j &=& j & \text{for } j \in \mathbb{Z}_3 \\ 1 + j &=& j + 1 & \text{for } j \in \mathbb{Z}_3 \backslash \{2\} \\ 1 + 2 &=& 0 & \\ 2 + 2 &=& 1 & \end{array}$$

and a commutative multiplication satisfying

$$\begin{array}{rcll} 0j &=& 0 & \text{for } j \in \mathbb{Z}_3 \\ 1j &=& j & \text{for } j \in \mathbb{Z}_3 \\ (2)(2) &=& 1, & \end{array}$$

is a field.

Exercise 1.13 Show $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ with a commutative addition satisfying

$$\begin{array}{rcll} 0 + j &=& j & \text{for } j \in \mathbb{Z}_4 \\ 1 + j &=& j + 1 & \text{for } j \in \mathbb{Z}_4 \backslash \{3\} \\ 1 + 3 &=& 0 & \\ 2 + 2 &=& 0 & \\ 2 + 3 &=& 1 & \\ 3 + 3 &=& 2 & \end{array}$$

and a commutative multiplication satisfying

$$\begin{array}{rcll} 0j &=& 0 & \text{for } j \in \mathbb{Z}_4 \\ 1j &=& j & \text{for } j \in \mathbb{Z}_4 \\ (2)(2) &=& 0 & \\ (2)(3) &=& 2 & \\ (3)(3) &=& 1, & \end{array}$$

is an additive group but not a field.

[3] In this array, the "+" symbol on the left denotes addition in $\mathbb{Z}_3$ and the "+" symbol on the right denotes addition of the corresponding elements in $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, the set of natural numbers including zero. A similar convention holds for our definition of addition in $\mathbb{Z}_4$.

Comparison to the real plane $\mathbb{R}^2$

You may have noted (and you should note now if you didn't) that in our definition/construction of $\mathbb{C}$, the symbol i "functions" in primarily two capacities within the set $\mathbb{C}$. First of all the symbol i behaves as a "placeholder" indicating that the two real numbers a and b in the complex number $a + bi$ are kept essentially separate. In this way, the real number b preceeding (or sometimes following as in ib) is very much like a component in the ordered pair (a, b) of real numbers. In fact, there is a one-to-one and onto function $\gamma : \mathbb{R}^2 \to \mathbb{C}$ given by

$$\gamma(x, y) = x + iy$$

which we will now consider briefly and will be extremely important for us later. With regard to the role of i as a placeholder, let us note that when a complex number is expressed as $z = a + bi$ with $a, b \in \mathbb{R}$, then the numbers a and b are called the **real part** and the **imaginary part** of the complex number $z \in \mathbb{C}$ respectively, and there is an associated notation:

$$\operatorname{Re} z = a, \qquad \operatorname{Im} z = b.$$

Exercise 1.14 Show

$$\operatorname{Re}(z + w) = \operatorname{Re}(z) + \operatorname{Re}(w) \qquad \text{for all } z, w \in \mathbb{C},$$

but

$$\operatorname{Re}(zw) \neq (\operatorname{Re} z)(\operatorname{Re} w) \qquad \text{for at least some } z, w \in \mathbb{C}.$$

Characterize the complex pairs $(z, w) \in \mathbb{C}^2$ for which $\operatorname{Re}(zw) = (\operatorname{Re} z)(\operatorname{Re} w)$.

The second "function" of the symbol i in the complex numbers is as a particular "algebraic variable" for which the peculiar condition $i^2 = -1$ holds. You might have noticed this in the formula for complex multiplication, and if you didn't, you should notice it now: If we treat i simply as an algebraic variable like the variable x in the polynomial $a + bx$, then

$$(a + bi)(c + di) = ac + (ad + bc)i + bd\ i^2.$$

You see, the factor $ad+bc$ multiplying the placeholder i, in other words the imaginary part of the product, agrees with our definition of complex multiplication. Furthermore, and most crucially, if we replace i^2 with -1, then the full definition of complex multiplication results. This latter "function" of i and the peculiar relation[4]

$$i^2 = -1 \tag{1.5}$$

is really what gives complex analysis its distinctive characteristics as a subject. It has been said that if someone has an idea comparable to this use of i as an algebraic symbol with this property, i.e., the idea of the construction of the complex numbers, then that person has done enough as a mathematician for one lifetime. The entire subject of complex analysis may be viewed as following out the consequences of the relation (1.5).

The set $\mathbb{R}^2$ is usually considered with a different algebraic structure than that of a field, but a field is involved. A **vector space** V over a field F is an additive group with a notion of **scaling** by elements from the field. That is, there is a scaling $F \times V \ni (\alpha, v) \mapsto \alpha v \in V$ satisfying certain properties. More precisely, the vector space V is required to be a commutative group under addition (that is, vector addition is associative, commutative, there is an additive identity (zero) vector, and there are additive inverses) and the scaling is required to satisfy the following four properties:

VS1 $\alpha(\beta v) = (\alpha\beta)v$ for $\alpha, \beta \in F$ and $v \in V$.

VS2 $1v = v$ for all $v \in V$ where 1 is the multiplicative identity in the field F.

VS3 $\alpha(v + w) = \alpha v + \alpha w$ for $\alpha \in F$ and $v, w \in V$.

VS4 $(\alpha + \beta)v = \alpha v + \beta v$ for $\alpha, \beta \in F$ and $v \in V$.

In this context, the field is called a **field of scalars**. The four properties can be named/described as follows:

VS1	associativity of scaling,
VS2	scaling by the field identity,
VS3	a scalar distributes across a vector sum,
VS4	a vector distributes across a sum of scalars.

The properties **VS3** and **VS4** are kinds of distributive properties (in addition to the distributive property required to hold for multiplication across addition in the field of scalars).

[4]It is of course from this relation that the symbol i gets is moniker as an "imaginary" number: The symbol i is the unthinkable solution to the polynomial equation $x^2 + 1 = 0$. (Once one thinks of this solution, the other solution is not so unthinkable.)

The set $\mathbb{R}^2$ is most naturally considered as a vector space over the field $\mathbb{R}$, and you are probably familiar with $\mathbb{R}^2$ considered as such. Let us review briefly:

- The addition in $\mathbb{R}^2$ is componentwise addition:
$$(x, y) + (a, b) = (x + a, y + b).$$
- The additive identity is the **zero vector** $(0, 0)$.
- The scaling in $\mathbb{R}^2$ is also componentwise:
$$\alpha(x, y) = (\alpha x, \alpha y).$$

This is the natural (algebraic) structure on $\mathbb{R}^2$, and generally I (like to and will) consider $\mathbb{R}^2$ in precisely this way.

Exercise 1.15 Show that $\mathbb{C}$ is a **real vector space** with the usual complex addition defined above and scaling given by
$$\alpha(a + bi) = \alpha a + \alpha bi$$
where $\alpha \in \mathbb{R}$, $a = \mathrm{Re}(a + bi)$ and $b = \mathrm{Im}(a + bi)$.

With the real vector space structure on $\mathbb{C}$ defined in Exercise 1.15, the mapping $\gamma : \mathbb{R}^2 \to \mathbb{C}$ by $\gamma(x, y) = x + yi$ is a **vector space isomorphism**, meaning not only is γ one-to-one and onto, but also linear:
$$\gamma(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\gamma(\mathbf{x}) + \beta\gamma(\mathbf{y}) \qquad \text{for } \alpha, \beta \in \mathbb{R} \text{ and } \mathbf{x}, \mathbf{y} \in \mathbb{R}^2.$$

This means that $\mathbb{R}^2$ and $\mathbb{C}$ are essentially the same real vector spaces, only the "look" of the vectors is changed. As suggested above, I am rather inclined to leave it (that is the discussion of comparing $\mathbb{R}^2$ and $\mathbb{C}$) at that: $\mathbb{R}^2$ and $\mathbb{C}$ are isomorphic as (two-dimensional) real vector spaces.

Given the introduction of $\mathbb{C}$ given by Brown and Churchill, however, I will say more.

Products on $\mathbb{R}^2$

One can introduce various **products** between pairs of vectors in the vector space $\mathbb{R}^2$. Two familiar products come to mind. First of all, there are **inner products**, the most familiar of which is the **Euclidean dot product**:

$$(x, y) \cdot (a, b) = ax + by. \tag{1.6}$$

This, it will be noted, is not a product operation in the sense that the value of the product is another vector in $\mathbb{R}^2$. The Euclidean product of two vectors in $\mathbb{R}^2$ is a scalar: $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$.

More generally, an **inner product** on $\mathbb{R}^2$ is a function

$$\langle \,\cdot\, , \,\cdot\, \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

satisfying the following properties:

IP1 $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$
An inner product is **symmetric**.

IP2 $\langle \alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z} \rangle = \alpha\langle \mathbf{x}, \mathbf{z} \rangle + \beta\langle \mathbf{y}, \mathbf{z} \rangle$ for $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$
An inner product is **bilinear**.

IP3 $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ for $\mathbf{x} \in \mathbb{R}^2$ with equality if and only if $\mathbf{x} = (0, 0)$.
An inner product is **positive definite**.

All inner products on $\mathbb{R}^2$ have the form

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot A\mathbf{y}$$

where A is a **positive definite** symmetric 2×2 matrix. If the element in the[5] i-th row and j-th column of the matrix A is a_{ij}, then symmetric means $a_{ji} = a_{ij}$. Positive definite means exactly the condition required by **IP3** which, it turns out in this case, is equivalent to the condition that there exists a basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ for $\mathbb{R}^2$ consisting of **eigenvectors** of A, that is with $A\mathbf{u}_j = \lambda_j \mathbf{u}_j$, for some positive (real) numbers λ_1 and λ_2.

One may also consider the **cross-product** of vectors in $\mathbb{R}^2$. The value of the cross-product can be thought of as a single number

$$\times((x, y), (a, b)) = bx - ay. \tag{1.7}$$

[5]When we use indices here in reference to matrices the symbol i does **not** represent $\sqrt{-1}$ but rather an element of $\{1, 2\}$.

Usually, however, the cross-product of two vectors in $\mathbb{R}^2$ is considered as a vector, but it it not a vector in $\mathbb{R}^2$, but rather in $\mathbb{R}^3$:

$$(x, y) \times (a, b) = (x, y, 0) \times (a, b, 0) = (0, 0, bx - ay).$$

Either way, the cross-product is not a product operation with values back in the vector space $\mathbb{R}^2$.

Having made these observations about the dot product and the cross-product of vectors in the real vector space $\mathbb{R}^2$, let us return to consideration of the one-to-one correspondence $\gamma : \mathbb{R}^2 \to \mathbb{C}$ given by

$$\gamma(x, y) = x + yi.$$

First of all, it is quite possible to "induce" a product of vectors in $\mathbb{R}^2$ by using the corresponding complex product in $\mathbb{C}$. Let us do that. We will say

$$\begin{aligned}(x, y)(a, b) &= \gamma^{-1}(\gamma(x, y)\gamma(a, b)) \\ &= \gamma^{-1}((x + yi)(a + bi)) \\ &= \gamma^{-1}(ax - by + (bx + ay)i) \\ &= (ax - by, bx + ay). \end{aligned} \tag{1.8}$$

This is an unusual and somewhat unnatural thing to do. I'm only doing it because this is, roughly speaking, what is done in BC. From the point of view taken above, the result is a "realization," or field isomorphic copy, of $\mathbb{C}$ in the form of $\mathbb{R}^2$.

Technically, Brown and Churchill suggest that $\mathbb{C}$ **is** $\mathbb{R}^2$ with the product defined by (1.8). Then one is left to execute the unnatural algebraic manipulations suggested by Exercise 2.1(b-c) in BC.

Exercise 1.16 Translate the algebraic expressions

(a) $(\sqrt{2} - i) - i(1 - \sqrt{2}i)$,

(b) $(2 - 3i)(-2 + i)$, and

(c) $(3 + i)(3 - i)[1/5 + (1/10)i]$

into the peculiar ordered pair notation of BC, simplify the expressions, and write the results in both forms, i.e., the weird/unnatural $\mathbb{R}^2$ form and the usual complex form.

Solution:

(a) $[(\sqrt{2},0)-(0,1)]-(0,1)[(1,0)-(0,\sqrt{2})]$.

$$\begin{aligned}[(\sqrt{2},0)-(0,1)]-(0,1)[(1,0)-(0,\sqrt{2})] &= (\sqrt{2},-1)-(0,1)(1,-\sqrt{2})\\ &= (\sqrt{2},-1)-(\sqrt{2},1)\\ &= (0,-2)\\ &= -2i.\end{aligned}$$

(b) $(2,-3)(-2,1)=(-1,8)=-1+8i$.

(c) $(3,1)(3,-1)(1/5,1/10)$. In principle, we already know multplication in $\mathbb{C}$ is associative (and commutative), so these products can be executed in any order. Just for "fun," I'll compute in both associative orders. The first product is interesting on its own:

$$(3+i)(3-i)=(3,1)(3,-1)=(10,0)=10.$$

$$(3,1)(3,-1)(1/5,1/10)=(10,0)(1/5,1/10)=(2,1)=2+i.$$

$$\begin{aligned}(3,1)[(3,-1)(1/5,1/10)] &= (3,1)(7/10,1/10)\\ &= (2,1)\\ &= 2+i.\end{aligned}$$

Of course, this won't kill anyone, but I'm not sure it will help anyone too much either.

The nominal advantage of inducing the full complex product structure on $\mathbb{R}^2$ is that there is a certain symmetry between the elements $1=(1,0)$ and $i=(0,1)$. When it comes to the complex product, however, these elements simply do not play symmetric roles—they behave anything but symmetrically—as can be easily seen by computation of the following complex products:

$$(1)(1)=(1,0)(1,0)=(1,0)=1$$

and

$$(i)(i)=(0,1)(0,1)=(-1,0)=-1.$$

Relation(s) among products

Since we have gone this far, let's try to express this strange complex product on $\mathbb{R}^2$ in terms of the Euclidean dot product and the cross-product mentioned above. For this discussion, let us consider the cross-product as a function $\times : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ and use the notation

$$(x, y) \times (a, b) = xb - ya. \tag{1.9}$$

Notice that while the dot product

$$(x, y) \cdot (a, b) = ax + by \tag{1.10}$$

is symmetric, the cross-product is **antisymmetric**:

$$(a, b) \times (x, y) = ay - bx = -(x, y) \times (a, b).$$

Let us call the two products of (1.10) and (1.9) the elementary geometric products on $\mathbb{R}^2$.

Comparing the expression

$$(x, y)(a, b) = (ax - by, bx + ay)$$

for the complex product to the expressions (1.10) and (1.9) suggests some relation(s) should be easy to obtain. Surprisingly, things are not entirely simple: This complex product is doing something geometrically which is not immediately expressible in terms of the elementary geometric products. A first few attempts at obtaining a relation suggests the introduction of certain natural **reflections** on $\mathbb{R}^2$. These are the linear functions

$\rho_+ : \mathbb{R}^2 \to \mathbb{R}^2$	by	$\rho_+(x, y) = (y, x)$,	reflection about the line $y = x$,
$\rho_- : \mathbb{R}^2 \to \mathbb{R}^2$	by	$\rho_+(x, y) = (-y, -x)$,	reflection about the line $y = -x$,
$\epsilon_1 : \mathbb{R}^2 \to \mathbb{R}^2$	by	$\epsilon_1(x, y) = (x, -y)$,	reflection about the x-axis, and
$\epsilon_2 : \mathbb{R}^2 \to \mathbb{R}^2$	by	$\epsilon_2(x, y) = (-x, y)$,	reflection about the y-axis.

It will be noted that another linear function is in play here: $\rho_- = \alpha \circ \rho_+$ and $\epsilon_2 = \alpha \circ \epsilon_1$ where

$\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ by $\alpha(x, y) = (-x, -y)$ is the **antipodal map**.

The reflection ϵ_1 about the x-axis will play a particularly important role for us later on and may also be referred to as the **conjugation map** on $\mathbb{R}^2$.

With these linear functions in hand, the "coordinates," i.e., real and imaginary parts of the complex product can be expressed in multiple ways. For example,

$$\begin{aligned} ax - by &= (a, -b) \cdot (x, y) \\ &= (a, b) \cdot (x, -y) \\ &= (-b, -a) \times (x, y) \\ &= (a, b) \times (y, x). \end{aligned}$$

Exercise 1.17 Rewrite one of the factors in each of the expressions above for the quantity $ax - by$ in terms of a linear map applied to one of the vectors (a, b) or (x, y). For example,

$$(a, -b) \cdot (x, y) = \epsilon_1(a, b) \cdot (x, y).$$

Find four expressions for $ay + bx$ having similar forms, so that for example

$$(a + bi)(x + yi) = \epsilon_1(a, b) \cdot (x, y) + \epsilon_1(a, b) \times (x, y)\ i. \tag{1.11}$$

Among the various forms the complex product $(a + bi)(x + iy)$ may take in terms of the elementary geometric products and linear maps of the vectors (a, b) and (x, y) in $\mathbb{R}^2$ as suggested by Exercise 1.17, certain forms like the one given in (1.11) display more (at least a little more) uniform appearance than others.

Exercise 1.18 Let us say an expression

$$(a + bi)(x + iy) = R_1(a, b) \otimes R_2(x, y) + R_3(a, b) \odot R_4(x, y)$$

for $(a + bi)(x + iy)$ obtained as in Exercise 1.17, where $R_j \in \{\rho_\pm, \epsilon_1, \epsilon_2, \text{id}\}$, j=1,2,3,4 and $\otimes$ and $\odot$ are elementary geometric products, gets a "uniformity point" if any one of these three properties hold:

(i) The same reflection appears in both the real and imaginary parts.

(ii) The same elementary product is used in both the real and imagainary parts.

(iii) The identity function $\text{id} : \mathbb{R}^2 \to \mathbb{R}^2$ appears in both the real and imaginary parts.

Show that no form gets three uniformity points and among the forms with two uniformity points are

$$\begin{aligned}(a+bi)(x+yi) &= \epsilon_1(a,b)\cdot(x,y)+\epsilon_1(a,b)\times(x,y)\ i\\ &= \epsilon_1(a,b)\cdot(x,y)+\rho_+(a,b)\cdot(x,y)\ i\\ &= (a,b)\cdot\epsilon_1(x,y)+(a,b)\cdot\rho_+(x,y)\ i\\ &= \rho_-(a,b)\times(x,y)+\epsilon_1(a,b)\times(x,y)\ i\\ &= (a,b)\times\rho_+(x,y)+(a,b)\times\epsilon_2(x,y)\ i\\ &= (a,b)\times\rho_+(x,y)+(a,b)\cdot\rho_+(x,y)\ i.\end{aligned}$$

Are there any more?

It is a little surprising (to me) that another natural linear transformation of $\mathbb{R}^2$ does not seem to come up "naturally" in this discussion. This linear function is sometimes called "perp," and it is also rotation counterclockwise by an angle $\pi/2$:

$$(x,y)^\perp = (-y,x).$$

Exercise 1.19 Can you find a nice/interesting expression for the complex product $(a+bi)(x+yi)$ using the orthogonal rotation applied to the vectors $\gamma^{-1}(a+bi)=(a,b)$ and $\gamma^{-1}(x+yi)=(x,y)$ in $\mathbb{R}^2$?

Further Discussion: I can think of three more possible motivations for introducing/thinking of the complex plane as the real plane $\mathbb{R}^2$ with the complex product $(a,b)(x,y)\mapsto(ax-by,ay+bx)$. The first is that this avoids the (otherwise important) identification of $\mathbb{R}^2$ with $\mathbb{C}$ as real vector spaces (and otherwise as fundamentally different algebraic sets). Perhaps Brown and Churchill felt the concept of a function $\gamma:\mathbb{R}^2\to\mathbb{C}$ was, somehow, too advanced.

The second is that this approach in BC makes the complex plane seem somehow less exotic and, in particular, gives the instructor a handy answer to the perhaps inevitable question: "What is the imaginary number i?" The answer from this point of view is: "The symbol i is nothing but a name for the element $(0,1)$." In a certain sense, this goes back to the symmetry between $(1,0)$ and $(0,1)$, which in fact and in principle is not present when one introduces the complex product on $\mathbb{R}^2$.

The third "advantage" of the strict identification of $\mathbb{C}$ with $\mathbb{R}^2$ is that addition of complex numbers is immediately translatable into vector addition of "points" in $\mathbb{R}^2$ as discussed in section 1.4 of BC. One might say also that this correspondence is slightly obscured with the complex notation. Take for example the expression of a vector/point $(x, y) \in \mathbb{R}^2$ in terms of the standard basis vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. Specifically, it appears that the decomposition

$$(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2$$

expresses relatively nontrivial and rather important content. In the complex plane, however, the corresponding relation $a+bi = a\cdot 1+b\cdot i$ becomes essentially invisible with the "standard basis vectors" given by $1 = 1 + 0 \cdot i$ and $i = 0 + 1\cdot$ usually not receiving special comment as such.

From the alternative point of view in which $\mathbb{R}^2$ is a well-known real vector space and $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ is a fundamentally different, and nominally exotic, algebraically complete[6] field, the "symbol" i is, as described above, both a placeholder and an algebraic symbol, commuting with itself and all real numbers (both additively and multiplicatively) and having the property that $i^2 = 1$. In short, $\mathbb{C}$ is a collection of symbols $a + bi$ to which the algebraic operations of addition and multiplication described above may be applied/executed. There is indeed perhaps a little mystery in this form of the field $\mathbb{C}$ at first, but most people become used to and comfortable with this representation of $\mathbb{C}$ rather quickly as I suspect is the case also with just about anyone who studies BC.

The **complex plane** is a geometric representation of the field $\mathbb{C}$, which though it looks rather like $\mathbb{R}^2$ and is isomorphic to $\mathbb{R}^2$ in certain respects, is fundamentally different in certain ways as well. Prominent among these differences is the labeling of points. In particular, the points along the vertical (or imaginary) axis in $\mathbb{C}$ are labeled bi rather than $(0, b)$.

Finally, the discussion above illustrates, perhaps, that the introduction of a complex product on $\mathbb{R}^2$ is somewhat unnatural and cumbersome. On the other hand, it may be objected that this product is no more nor less cumbersome than the complex product in $\mathbb{C}$.

Exercise 1.20 We have given above a general definition of a **vector space over a field**. Let me repeat that definition here: A set V is a **vector space** over the field F if there is an operation of addition $+ : V \times V \to V$ and a scaling $F \times V \to V$ satisfying the following properties:

VSadd1 $(v + w) + z = v + (w + z)$ for all $v, w, z \in V$.

[6]That is, every polynomial of degree $m \geq 1$ with coefficients in $\mathbb{C}$ factors in the from $c(z-z_1)(z-z_2)\cdots(z-z_m)$ where $z_1, z_2, \ldots, z_m \in \mathbb{C}$. The polynomial $x^2 + 1$ with real coefficients does not have a similar factorization with respect to the real field.

VSadd2 There is a **zero vector** $\mathbf{0} \in V$ for which

$$\mathbf{0} + v = v + \mathbf{0} = v \qquad \text{for all } v \in V.$$

VSadd3 For each $v \in V$, there is an additive inverse $-v \in V$ for which

$$(-v) + v = v + (-v) = \mathbf{0}.$$

VSadd4 $v + w = w + v$ for all $v, w \in V$.

VS1 $(\alpha\beta)v = \alpha(\beta v)$ for all $\alpha, \beta \in F$ and $v \in V$.

VS2 $1v = v$ for all $v \in V$ where $1 \in F$ is the multiplicative identity in the field.

VS3 $\alpha(v + w) = \alpha v + \alpha w$ for all $\alpha \in F$ and $v, w \in V$.

VS4 $(\alpha + \beta)v = \alpha v + \beta v$ for all $\alpha, \beta \in F$ and $v \in V$.

Two possible choices for the field are $\mathbb{R}$ and $\mathbb{C}$. If the field is taken to be $\mathbb{R}$, the vector space is called a **real vector space**. If the field is taken to be $\mathbb{C}$, the vector space is called a **complex vector space**. The standard finite dimensional examples of the former are the spaces $\mathbb{R}^n$ for $n = 1, 2, 3, \ldots$, and the standard examples of the latter are the spaces

$$\mathbb{C}^n = \{(z_1, z_2, \ldots, z_n) : z_1, z_2, \ldots, z_n \in \mathbb{C}\}.$$

Here is the exercise: Show the vector space W is a complex vector space where, as a set, $W = \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$, the addition is componentwise

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

as usual, and the scaling $\mathbb{C} \times V \to V$ is given by

$$(a + bi)(x, y) = (ax - by, ay + bx).$$

Geometric interpretation of the complex product

There is a nice geometric interpretation of the complex product $(a + bi)(x + yi)$ in terms of the vectors $\gamma^{-1}(a + bi) = (a, b)$ and $\gamma^{-1}(x + yi) = (x, y)$ in $\mathbb{R}^2$, and it is very very important in complex analysis, but we must go in rather a different direction to find it. In particular, we must fundamentally turn away from the idea of symmetry and any suggestion of a symmetry between $1 \in \mathbb{C}$ and $i \in \mathbb{C}$.

Consider the first factor $a+bi \in \mathbb{C}$ and its real counterpart $(a,b) \in \mathbb{R}^2$. If $a+bi \neq 0$, then $\sqrt{a^2+b^2} > 0$, and

$$\mathbf{u} = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \right)$$

is a point on the unit circle $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\} \subset \mathbb{R}^2$. This is an extremely important observation. In particular, such a point determines a unique **argument**, that is, the angle θ between the segment connecting $(0,0)$ to (a,b) and the positive x-axis. We have to be a little careful when we say the argument is unique. The angle θ determined by the relation

$$\mathbf{u} = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \right) = (\cos\theta, \sin\theta) \tag{1.12}$$

is not entirely unique of course; it is unique up to an additive (integer) multiple of 2π:

$$\theta = \theta_0 + 2\pi k, \qquad k \in \mathbb{Z} = \{0, \pm1, \pm2, \pm3, \ldots\},$$

where $\theta = \theta_0$ is any particular angle satisfying (1.12). There is a unique choice of θ_0 in any half open interval of length 2π like $(-\pi, \pi]$ or $[0, 2\pi)$. What I have just described is extremely important to understand.

Now given an angle, that is an argument, measured either from the positive x-axis in $\mathbb{R}^2$ or from the positive real axis in $\mathbb{C}$, there is an associated rotation of the space counterclockwise by the angle θ. The rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$R_\theta(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

And this, I think, makes rather clear the geometric interpretation of the product

$$(a+bi)(x+yi) = xa - yb + (xb+ya)i \in \mathbb{C}. \tag{1.13}$$

Specifically, taking the argument θ defined by (1.12) and determined (very asymmetrically) by the first factor $a+bi$, the product in (1.13) can be written immediately as

$$(a+bi)(x+yi) = \sqrt{a^2+b^2}[x\cos\theta - y\sin\theta + (x\sin\theta + y\cos\theta)i].$$

That is, up to homogeneous scaling by the factor $\sqrt{a^2+b^2}$, the Euclidean length of the vector $(a,b) = \gamma^{-1}(a+bi) \in \mathbb{R}^2$, the complex product has

> real part the x-component of the counterclockwise rotation $R_\theta(x,y)$, and
> imaginary part the y-component of the counterclockwise rotation $R_\theta(x,y)$.

In terms of the vector space isomorphism $\gamma : \mathbb{R}^2 \to \mathbb{C}$:

$$(a + bi)(x + yi) = \sqrt{a^2 + b^2}\ \gamma^{-1}(\ R_\theta \circ \gamma(x + yi)\ ). \tag{1.14}$$

To say it a third (!) way: The complex product of $a + bi$ and $x + yi$ corresponds to rotating the real vector (x, y) in $\mathbb{R}^2$ counterclockwise by the argument θ of $a + bi$ and scaling the result by the Euclidean length/norm of the corresponding real vector (a, b). In the complex plane/field the value $\sqrt{a^2 + b^2}$ is called, simply, the **absolute value** of the complex number $a + bi$, but we will come (back) to this topic in more detail later.

Exercise 1.21 Given an argument θ, express the rotation R_θ of $\mathbb{R}^2$ counterclockwise by the angle θ in terms of matrix multiplication (as used for example in linear algebra).

Additional "follow up" notes

The material in my section 1 above, as suggested by the title, corresponds roughly to the material in sections 1 and 2 of BC Chapter 1.

Expressions using "set brackets and colons" like the ones appearing in (1.1) and (1.2) are called "set specifications." Hopefully, you have a basic idea of what is being expressed in a set specification. If not, at least a first good step is to read such an expression correctly. The right side of (1.2) for example should be (formally) read something like this:

The set of all ordered pairs **such that** each component is a real number.

Every time you read a set specification, you can think, the opening curly bracket is read "the set of all" and the colon may be read "such that." Once you get the hang of it and internalize the meaning of such expressions, you can vary the reading. For example,

$$\{x \in \mathbb{R} : x > 5\}$$

can be read "The set of real numbers greater than 5," that is to say "the open interval in $\mathbb{R}$ from 5 to $+\infty$."

Brown and Churchill's complex product on $\mathbb{R}^2$ seems somewhat unnatural and cumbersome. On the other hand, it really is some form of the usual complex product which many consider rather natural, at least in $\mathbb{C}$. In fact, everyone who learns complex analysis should eventually get used to this product on $\mathbb{C}$ and consider it relatively natural, at least be able to manipulate it in various ways, and probably even find it kind of elegant. On the third (!) hand, most people find the complex product somewhat cumbersome at first however it is introduced, so if you do, just be patient and keep working on it.

1.2 BC Section 1.3

Carefully solving a simple equation

Let $\alpha \in \mathbb{C}\backslash\{0\}$ and assume

$$\alpha z = 0. \tag{1.15}$$

If equation (1.15) holds for some $z \in \mathbb{C}$, then since α has a multiplicative inverse $\alpha^{-1} \in \mathbb{C}$ we know

$$z = (\alpha^{-1}\alpha)z = \alpha^{-1}(\alpha z) = \alpha^{-1}(0). \tag{1.16}$$

Exercise 1.22 Prove carefully that the product $(w)(0) = 0$ for any $w \in \mathbb{C}$.

Solution: $(w)(0) = w(0 + 0) = (w)(0) + (w)(0)$. Therefore,

$$\begin{aligned}(w)(0) - (w)(0) &= [(w)(0) + (w)(0)] - (w)(0)\\ &= (w)(0) + [(w)(0) - (w)(0)]\\ &= (w)(0) + 0\\ &= (w)(0).\end{aligned}$$

Thus, $(w)(0) = (w)(0) - (w)(0) = 0$. □

Applying Exercise 1.22 to (1.16) we find $z = 0$. We have established the following (not so surprising) result:

Theorem 1 *If $\alpha z = 0$ with $\alpha \in \mathbb{C}\backslash\{0\}$ and $z \in \mathbb{C}$, then $z = 0 \in \mathbb{C}$.*

The following exercises give some useful assertions equivalent to the assertion of Theroem 1

Exercise 1.23 Use Theorem 1 to show the following:

If $zw = 0$ for $z, w \in \mathbb{C}$, then either $z = 0$ or $w = 0$.

Note that the conclusion here allows the possibility $z = w = 0$.

Exercise 1.24 Use Theorem 1 to show the following:

If $z, w \in \mathbb{C}\backslash\{0\}$, then either $zw \neq 0$.

In our solution of Exercise 1.21 we "subtracted" the complex number $(w)(0)$ from both sides of an equation. If we wanted to be even more careful (and more "axiomatically correct") we would first **define subtraction** in terms of the algebraic properties discussed previously. That is, if $z, w \in \mathbb{C}$, then w has an additive inverse $-w \in \mathbb{C}$ and we define

$$z - w = z + (-w).$$

Exercise 1.25 (displayed equation (3) on page 6 in BC) Given the complex numbers $z = x + yi$ and $w = a + bi$, find the real and imaginary parts of $w - z$ (and carefully justify your "computation.")

Similarly, there is a nominal difference between the "division" z/w where $w \in \mathbb{C}\backslash\{0\}$ and "multiplication by the multiplicative inverse" zw^{-1}.

Exercise 1.26 (displayed equation (3) on page 6 in BC) When we discussed axiom 7 for the real numbers (the existence of multiplicative inverses) we were a bit sloppy on the distinction between division and multiplication by the multiplicative inverse. How should our discussion of this axiom be modified to become more axiomatically correct?

If you're really into this sort of thing, here are a(nother) couple exercises for you:

Exercise 1.27 **(a)** Explain the difference between the complex number bi and the complex product $(i)(b)$. Decide which expression arises first from an axiom defining the complex numbers and prove the other one is equal to the first one.

(b) Give a careful solution of the simple equation $\alpha z = 1$ when $\alpha \in \mathbb{C}\backslash\{0\}$.

There follows in BC a sequence of elementary, important, and somewhat tedious computations/identities. The most interesting is perhaps

$$\frac{z}{w} = \frac{z\overline{w}}{|w|^2} \quad \text{or} \quad \frac{x+iy}{a+bi} = \frac{(x+iy)(a-bi)}{(a+bi)(a-bi)} = \frac{ax+by}{a^2+b^2} + \frac{ay-bx}{a^2+b^2}\, i \tag{1.17}$$

though the accompanying notation (and explanation) is not given. With anticipation of some repetition later, let us note the following: If $z = x + iy$ with $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$ as usual,

$\overline{z} = x - iy$ is the complex **conjugate** of z and

$|z| = \sqrt{x^2 + y^2}$ is the complex **modulus** (or absolute value) of z.

These are just definitions. With these definitions it is straightforward to verify everything in (1.17).

Let's consider one more tedious observation: According to BC "it can be shown that"

$$\left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right) = \frac{1}{z_1 z_2} \qquad \text{for } z_1, z_2 \in \mathbb{C}\backslash\{0\}.$$

What this means (in words) is that "the product of the (unique) multiplicative inverses of z_1 and z_2 is the (unique) multiplicative inverse of the product $z_1 z_2$." In order to "show" this astounding assertion we need to (ever so carefully) compute the product(s)

$$\left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right) z_1 z_2 \qquad \left(\quad \text{and } z_1 z_2 \left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right) \quad \right)$$

obtaining (if everything goes okay) the multiplicative identity $1 \in \mathbb{C}$. Let's see if we can do it:

$$\begin{aligned}
\left(\tfrac{1}{z_1}\right)\left(\tfrac{1}{z_2}\right) z_1 z_2 &= \left(\tfrac{1}{z_1}\right)\left(\tfrac{1}{z_2}\right) z_2 z_1 && \text{(commutativity)} \\
&= \left(\tfrac{1}{z_1}\right)\left[\left(\tfrac{1}{z_2}\right) z_2\right] z_1 && \text{(associativity)} \\
&= \left(\tfrac{1}{z_1}\right)[1]\, z_1 && (1/z_2 = z_2^{-1}) \\
&= \left(\tfrac{1}{z_1}\right) z_1 && (1 \text{ is } 1 \in \mathbb{C}) \\
&= 1 && (1/z_1 = z_1^{-1}).
\end{aligned}$$

We now know $(1/z_1)(1/z_2) = 1/(z_1 z_2)$. Q.E.D.

Welcome to the exciting world of (complex) arithmetic.

The binomial theorem

The last observation of section 1.3 of BC is also familiar (and pretty easy) but it is also very important and involves a couple constructions that may be worth reviewing.

If $z, w \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, then

$$\begin{aligned}
(z+s)^n &= \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} z^{n-k} w^k
\end{aligned}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is the "combination" of n things taken k at a time,[7] also suggestively known in some circles as the **binomial coefficient**.

Exercise 1.28 Prove the binomial theorem for complex numbers using induction (and the arithmetic axioms).

Exercise 1.29 How many ways are there to choose k objects from among n **identical** objects?

1.3 Some geometry of complex numbers

Here are my notes on sections 1.4-5 of BC.

1.3.1 Addition of complex numbers

The **sum** $z + w$ of complex numbers z and w corresponds to the diagonal (through the origin $0 \in \mathbb{C}$) of the parallelogram

$$Q = \{sz + tw : 0 \leq s, t \leq 1\}$$

determined by the complex numbers; see Figure 1.1. The difference $z - w$ corresponds, roughly speaking, to the other diagonal of Q. To be more technically accurate, we should remember that $z - w$ is the sum of z and the additive inverse of w, so really, $z - w$ corresponds to the diagonal (through the origin) of a translation

$$Q_{-} = \{sz - tw : 0 \leq s, t \leq 1\}$$

of Q as illustrated in Figure 1.2. It is natural, of course, to think in terms of the familiar "vector" geometry in which $z - w$ corresponds to the diagonal in Q "pointing from w to z," i.e., the "vector" which when added/concatenated to w lands you at z.

[7] That is, the number of ways k objects can be chosen from among a set of n distinct objects (where the order of the choosing does not matter).

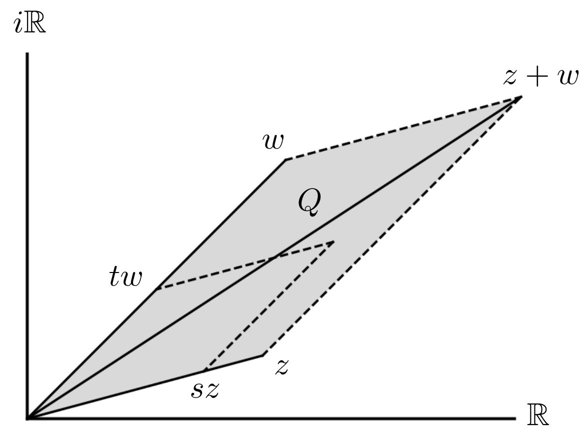

Figure 1.1: The parallelogram Q associated with the complex numbers z and w and the sum $z+w$.

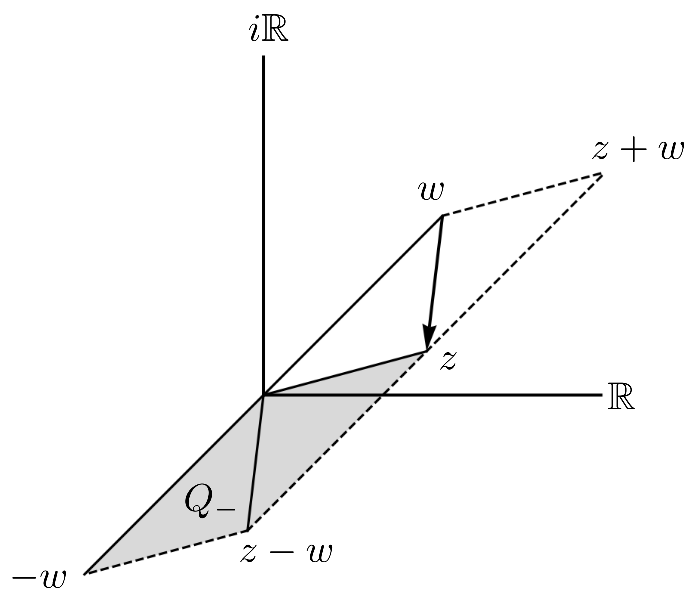

Figure 1.2: The parallelogram Q_- associated with the complex numbers z and w and the difference $z-w$. We have also drawn the "vector" corresponding to $z-w$ which points from w to z.

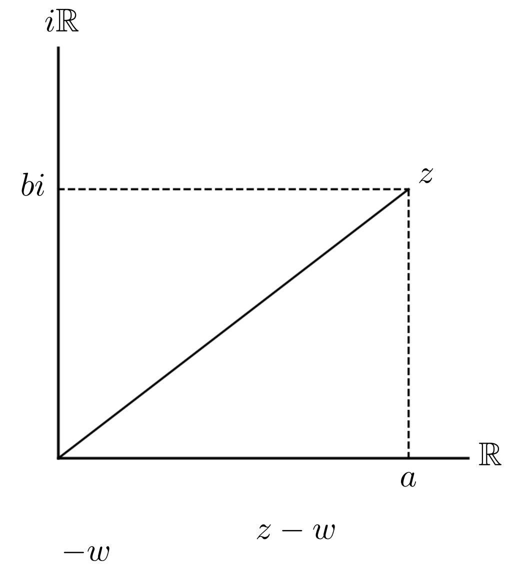

Figure 1.3: The modulus of a complex number $z = a + bi$.

In short, addition and subtraction in $\mathbb{C}$ are geometrically isomorphic to vector addition and subtraction in $\mathbb{R}^2$ (using the canonical correspondence $\gamma : \mathbb{R}^2 \to \mathbb{C}$ discussed in section 1.1 above.) Also, the **modulus** of $z = a + bi$ is given by

$$|z| = \sqrt{a^2 + b^2}$$

and corresponds to the length of the segment from 0 to z, or alternatively the segment in $\mathbb{R}^2$ connecting $\gamma^{-1}(0) = (0, 0)$ to $\gamma^{-1}(z) = (a, b)$.

The triangle inequality

$$|z + w| \le |z| + |w|$$

holds in $\mathbb{C}$.

Exercise 1.30 Draw an illustration “showing” the validity of the triangle inequality for the complex modulus, and determine from the illustration the condition(s) under which equality holds.

The observations above allow us to adopt certain geometric notation and terminology from $\mathbb{R}^2$. Given $\mathbf{p} \in \mathbb{R}^2$ and $r > 0$, the **ball** of radius r with center $\mathbf{p}$ is

$$B_r(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x} - \mathbf{p}\| < r\}$$

where

$$\|\mathbf{x}-\mathbf{p}\| = \sqrt{(\mathbf{x}-\mathbf{p})\cdot(\mathbf{x}-\mathbf{p})} = \sqrt{(x_1-p_1)^2+(x_2-p_2)^2}$$

in terms of the Euclidean norm and the Euclidean dot product applied to the difference of vectors $\mathbf{x} = (x_1, x_2)$ and $\mathbf{p} = (p_1, p_2)$ in $\mathbb{R}^2$. The **boundary** of $B_r(\mathbf{p})$ is the circle

$$\partial B_r(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}-\mathbf{p}\| = r\}.$$

In particular, $\mathbb{S}^1 = \partial B_1(0,0)$ is the **unit circle** in $\mathbb{R}^2$. There is perhaps no standard notation for the corresponding geometric objects in $\mathbb{C}$. Many authors write

$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

for the ball, i.e., disk, centered at $z_0 \in \mathbb{C}$ with radius $r > 0$. When I am working with $\mathbb{C}$, I am inclined to just use the (admittedly ambiguous) notation $B_r(z_0)$, $\partial B_r(z_0)$, $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ from $\mathbb{R}^2$ and let the context determine the meaning.

It is also perfectly natural to consider other geometric objects, lines, segments, ellipses, parabolas, etc., in $\mathbb{C}$. For example, (Example 3, page 10 of BC)

$$\{z \in \mathbb{C} : |z-4| + |z+4| = 10\}$$

is the set of $z = a + bi \in \mathbb{C}$ for which

$$\sqrt{(a-4)^2+b^2} + \sqrt{(a+4)^2+b^2} = 10.$$

That is,

$$2a^2 + 32 + 2b^2 + 2\sqrt{(a^2-16)^2 + (2a^2+32)b^2 + b^4} = 100$$

or

$$(a^2-16)^2 + (2a^2+32)b^2 + b^4 = (34 - a^2 - b^2)^2$$

or

$$-32a^2 + (16)^2 + (2a^2+32)b^2 = (34)^2 - 68(a^2+b^2) + 2a^2b^2$$

or

$$9a^2 + 25b^2 = (17-8)(17+8) = (9)(25)$$

or

$$\frac{a^2}{25} + \frac{b^2}{9} = 1.$$

Notice that in the "real form" of the equation of an ellipse, the lengths of the minor and major semiaxes play a priminent role, while in the complex form the focal length $c = 4$ is immediately visible along with the length 10 of the major axis.

This is a good time to return to our comparison of $\mathbb{C}$ to $\mathbb{R}$ as fields and point out two more properties of $\mathbb{R}$, one of which $\mathbb{C}$ shares and one of which $\mathbb{C}$ does not have. Let's start with the second property:

$\mathbb{R}$ is an **ordered field**. Given $a, b \in \mathbb{R}$, exactly one of the following holds:

$$a < b,$$
$$a = b, or$$
$$b < a.$$

There is no natural order relation on $\mathbb{C}$. It makes no sense, in general, to write $z < w$ or $z > w$ for $z, w \in \mathbb{C}$. If we ever write $z < w$ or $z > w$ (or even $z \leq w$ or $z \geq w$) for $z, w \in \mathbb{C}$, then we (must) mean z **and** w **are already known to satisfy** $z, w \in \mathbb{R} \subset \mathbb{C}$, and we are using the ordering on $\mathbb{R}$. We can of course consider the condition

$$|z| < |w| \qquad \text{for } z, w \in \mathbb{C}.$$

This is because $|z|, |w| \in \mathbb{R}$ and we can (again) use the ordering on $\mathbb{R}$. Incidentally, the real vector space $\mathbb{R}^2$, like the field $\mathbb{C}$, is not ordered.

All these sets $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{R}^2$ are **metric spaces**. That is, there exists a **distance function**

$$d : X \times X \to [0, \infty)$$

satisfying

MD1 $d(x, y) = d(y, x)$ for $x, y \in X$,

MD2 If $d(x, y) = 0$, then $x = y$, and

MD3 $d(x, z) \leq d(x, y) + d(y, z)$ for $x, y, z \in X$.

A set X with such a function $d : X \times X \to [0, \infty)$ defined on it is said to be a metric space, and the function d is called a **metric distance**. A metric distance is said to be symmetric **(MD1)**, positive definite **(MD2)**, and to satisfy a (metric) **triangle inequality (MD3)**. Notice that a set X need not be a vector space in order to be a metric space, though the three examples given by $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{R}^2$ all happen to be real vector spaces.

The metric distances on $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{R}^2$ are

$$|a - b| \qquad \text{(absolute value of the difference)},$$

$$|w - z| \qquad \text{(modulus of the difference), and}$$

$$\|(y_1, y_2) - (x_1, x_2)\| = \sqrt{(y_1 - x_1)^2 + (y_2 = x_2)^2} \qquad \text{(Euclidean norm of the difference)}$$

respectively.

Exercise 1.31 Show that for complex numbers, the metric triangle inequality

$$|z - w| \leq |z - \zeta| + |\zeta - w| \qquad \text{for } z, w, \zeta \in \mathbb{C} \tag{1.18}$$

implies the complex (field) triangle inequality

$$|z + w| \leq |z| + |w| \qquad \text{for } z, w \in \mathbb{C}. \tag{1.19}$$

Exercise 1.32 Show the condition (1.19) implies the condition (1.18) in $\mathbb{C}$ as well. Thus, the metric triangle inequality and the field triangle inequality are equivalent.

Brown and Churchill postpone the proof of the complex triangle inequality and give it as an exercise in a later section. Also, they seem to ignore the case of equality. These are not difficult. Since $|z + w|$ and $|z| + |w|$ are nonnegative quantities the triangle inequality is equivalent to the condition

$$|z + w|^2 \leq (|z| + |w|)^2 = |z|^2 + 2|z||w| + |w|^2.$$

On the other hand,

$$|z + w|^2 = (z + w)\overline{(z + w)} = (z + w)(\overline{z} + \overline{w}) = |z|^2 + z\overline{w} + w\overline{z} + |w|^2.$$

Therefore, the triangle inequality is equivalent to the condition

$$z\overline{w} + w\overline{z} \leq 2|z||w|.$$

Finally,

$$z\overline{w} + w\overline{z} = z\overline{w}\ \overline{(z\overline{w})} = 2\,\mathrm{Re}(z\overline{w}),$$

and $\mathrm{Re}(\zeta) \leq |\zeta|$ for any $\zeta \in \mathbb{C}$. In particular,

$$\mathrm{Re}(z\overline{w}) \leq |z\overline{w}| = |z|\ |\overline{w}| = |z|\ |w| = |zw|. \tag{1.20}$$

Exercise 1.33 Work backwards from (1.20) to establish (1.19).

In order for you to follow all the details of the argument above easily, perhaps some additional exercises would be helpful.

Exercise 1.34 Show that given any complex number $z = a + bi$ with $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$, one has

$$\mathrm{Re}(z) \le |\,\mathrm{Re}(z)| \le |z|$$

and

$$\mathrm{Im}(z) \le |\,\mathrm{Im}(z)| \le |z|.$$

Exercise 1.35 Show that given any complex numbers $z = a + bi$ with $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$ and $w = x + yi$ with $x = \mathrm{Re}(w)$ and $y = \mathrm{Im}(w)$, one has

$$|zw| = |z|\ |w|.$$

Exercise 1.36 Show that given any complex number $z = a + bi$ with $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$ one has

$$|\overline{z}| = |z|.$$

Let's consider the case of equality. Retracing the steps of Exercise 1.33 above, we see the condition $|z + w| = |z| + |w|$ is equivalent to

$$\mathrm{Re}(z\overline{w}) = |z\overline{w}|. \tag{1.21}$$

Given any complex number $\zeta = a + bi$, the condition $\mathrm{Re}(\zeta) = |\zeta|$ is

$$a = \sqrt{a^2 + b^2}.$$

Squaring both sides gives immediately that $b = 0$. Thus, $\zeta = a \in \mathbb{R}$ and furthermore, $a = |\zeta| \ge 0$. Thus we conclude that if ζ satisfies $\mathrm{Re}(\zeta) = |\zeta|$, then ζ must be a nonnegative real number. Applying this observation to the condition (1.21) for equality in the triangle inequality we find equality holds in the triangle inequality if and only if

$$z\overline{w} = \alpha \ge 0. \tag{1.22}$$

Remember that when we write an inequality like (1.22) involving a complex number, then we are asserting the complex number is real (and nonnegative). The usual way to interpret condition (1.22) is in terms of cases as follows:

> Equality holds in the triangle inequality if and only if
>
> **(i)** $z = 0$ or $w = 0$ or
>
> **(ii)** there is some $a > 0$ for which $z = aw$.

Exercise 1.37 Show the characterization of equality in the triangle inequality given by (1.22) is indeed equivalent to the condition that **(i)** or **(ii)** holds.

1.4 An estimate

The remainder of section 5 in BC is devoted to showing a certain inequality for polynomials expressing the fact that the modulus of a polynomial $P(z)$ must "grow" at infinity, i.e., as the modulus of the argument z tends to infinity. This will be used later (much later) in the proof of the fundamental theorem of algebra.

Theorem 2 (Example 3 of section 1.5 in BC) If P is a polynomial of order $n \geq 1$ with complex coefficients $a_0, a_1, \ldots, a_n$ so that

$$P(z) = \sum_{j=0}^{n} a_j z^j,$$

then there is some $R > 0$ for which

$$\frac{1}{|P(z)|} < \frac{2}{|a_n|R^n} \qquad \text{for} \qquad |z| > R. \tag{1.23}$$

It is implicit in the estimate (1.23) that $P(z) \neq 0$ for $|z| > R$.

Proof: Let's consider first the case $n = 1$ and assume $|z| > R_1$ (where R_1 is some positive number). In this case

$$P(z) = a_0 + a_1 z$$

with $a_1 \neq 0$. We wish to get an estimate on $|P(z)|$ from below. Note that

$$|a_1|R_1 < |a_1|\ |z| = |a_1 z| = |P(z) - a_0| \leq |P(z)| + |a_0| \tag{1.24}$$

by the triangle inequality. This gives us an estimate on $|P(z)|$ from below:

$$|P(z)| > |a_1|R_1 - |a_0| = |a_1| \left(R_1 - \frac{|a_0|}{|a_1|} \right). \tag{1.25}$$

By making R_1 larger than the fixed nonnegative number $|a_0|/|a_1|$ we can ensure $|P(z)| > 0$, and in particular does not vanish.

Corollary 1 If $P(z) = a_1 z + a_0$ with $a_1 \neq 0$, then there is some $R_0 > 0$ for which

$$|P(z)| > 0 \qquad \text{for} \qquad |z| > R_0. \tag{1.26}$$

Exercise 1.38 Generalize Corollary 1 to obtain the estimate (1.26) for any polynomial P of degree $n \geq 1$.

The actual estimate we want to get has the form

$$|P(z)| > \alpha |a_1| R$$

for some factor $\alpha > 0$, instead of zero on the right as in (1.26). Looking at (1.25) it is clear we are not going to get $\alpha = 1$. Fortunately, the proportion of R_1 we do get becomes larger when R_1 becomes larger. To see this observe that if we take $R_1 > 2|a_0|/|a_1|$, then

$$\frac{|a_0|}{|a_1|} < \frac{1}{2}R_1 \qquad \text{or} \qquad R_1 - \frac{|a_0|}{|a_1|} > R_1 - \frac{1}{2}R_1 = \frac{1}{2}R_1,$$

and we can conclude from (1.25) that for $|z| > R_1$ there holds

$$|P(z)| > \frac{1}{2}R_1|a_1|.$$

This gives us the desired estimate in the case $n = 1$ if we take $R = R_1$:

$$\frac{1}{|P(z)|} < \frac{2}{|a_1|R}.$$

Exercise 1.39 How large do you need to take R (in the case $n = 1$) to conclude

$$\frac{1}{|P(z)|} < \frac{3}{2|a_1|R} \qquad \text{for} \qquad |z| > R?$$

For $n \geq 2$, let us assume $R > 1$ so that

$$|z|^j \leq |z|^{n-1} \qquad \text{for} \qquad |z| > R \quad \text{and} \quad 0 \leq j \leq n-1.$$

Then following (1.24) we estimate as follows:

$$\begin{aligned} |a_n||z|^n &= |a_n z^n| \\ &= \left| P(z) - \sum_{j=0}^{n-1} a_j z^j \right| \\ &\leq |P(z)| + \left| \sum_{j=0}^{n-1} a_j z^j \right| \\ &\leq |P(z)| + \sum_{j=0}^{n-1} |a_j||z|^j. \end{aligned}$$

Our estimate from below then becomes

$$\begin{aligned}|P(z)| &\geq |a_n||z|^n - \sum_{j=0}^{n-1} |a_j||z|^j \\ &\geq |a_n||z|^{n-1}\left(|z| - \sum_{j=0}^{n-1} \frac{|a_j|}{|a_n|}\frac{|z|^j}{|z|^{n-1}}\right) \\ &> |a_n|R^{n-1}\left(R - \sum_{j=0}^{n-1} \frac{|a_j|}{|a_n|}\right).\end{aligned}$$

Taking

$$R > \max\left\{1, 2\sum_{j=0}^{n-1} \frac{|a_j|}{|a_n|}\right\}$$

we can say

$$\sum_{j=0}^{n-1} \frac{|a_j|}{|a_n|} < \frac{1}{2}R$$

and

$$|P(z)| > \frac{1}{2}|a_n|R^n.$$

The estimate

$$\frac{1}{|P(z)|} < \frac{2}{|a_n|R^n} \qquad \text{for} \qquad |z| > R$$

follows. □

Corollary 2 If $P : \mathbb{C} \to \mathbb{C}$ is a nonconstant polynomial, then there is some $r > 0$ and some $M > 0$ for which

$$|P(z)| > r \qquad \text{for} \qquad |z| > M.$$

In particular, $|P(z)| > 0$ for $|z| > M$, so P has no zeros exterior to $B_M(0)$.

1.5 Complex conjugates

This is section 1.6 in BC. We've already discussed/defined the complex conjugate of $z \in \mathbb{C}$ by

$$\overline{z} = \text{Re}(z) - i\,\text{Im}(z).$$

The "operation" of conjugation is very versatile. Primarily, one needs to check the properties and then remember them. If you do that, you will have obtained a rather important and powerful computational tool in complex analysis.

Exercise 1.40 Verify (and remember) the following properties of complex conjugation:

(a) $\overline{\overline{z}} = z$.

(b) $|\overline{z}| = |z|$.

(c) $\overline{z + w} = \overline{z} + \overline{w}$.

(d) $\overline{zw} = \overline{z}\,\overline{w}$.

(e) $z + \overline{z} = 2\,\mathrm{Re}(z)$.

(f) $z - \overline{z} = 2i\,\mathrm{Im}(z)$.

(g) $|z|^2 = z\overline{z}$.

(h) $z \in \mathbb{R}$ if and only if $\overline{z} = z$.

1.6 Elementary roots and powers

In sections 1.7-11 of BC, a discussion is given of powers and roots of complex numbers based on the exponential or "polar" form

$$z = |z|e^{i\theta} = |z|(\cos\theta + i\sin\theta)$$

where $\theta = \arg(z)$ is the argument of z as discussed above. In this discussion, the exponential notation is used "formally," that is to say somewhat non-rigorously as simply a notation for the complex expression $\cos\theta + i\sin\theta$. Thus, one should verify the usual exponential rules (or at least many/most of them) hold directly from this formal notational definition.

Recall that from our discussion above, every nonzero complex number determines a well-defined argument θ by

$$\begin{cases} \cos\theta = \mathrm{Re}\left(\dfrac{z}{|z|}\right) \\[2ex] \sin\theta = \mathrm{Im}\left(\dfrac{z}{|z|}\right). \end{cases} \tag{1.27}$$

Thus,

$$z = |z| \frac{z}{|z|} = |z| \left[\text{Re}\left(\frac{z}{|z|}\right) + i \ \text{Im}\left(\frac{z}{|z|}\right) \right] = |z|(\cos\theta + i\sin\theta).$$

Now, as mentioned above, BC introduces the relation

$$\cos\theta + i\sin\theta = e^{i\theta} \tag{1.28}$$

simply as notation. This is called **Euler's formula**. It should be noted that one can (perhaps) give a definition of e^w where w is a complex number independently and then "derive" Euler's formula instead of simply introducing notation.

Recall also that the **argument** θ is unique in the sense that there is a unique value of θ determined by (1.27) in any half open interval of length 2π. It is specified in BC that the **principal argument** is that value of θ in the interval $-\pi < \theta \le \pi$ and this particular value is denoted by

$$\text{Arg}(z)$$

(with a capital "A." The use of $\arg(z)$ and $\text{Arg}(z)$ is quite common, but the choice of the interval $-\pi < \theta \le \pi$ for the principal value of the argument is not a universal. Sometimes the principal argument is taken with $0 \le \theta < 2\pi$. In any case, we should know that when we see $\text{Arg}(z)$ in BC, the unique argument with $-\pi < \theta \le \pi$ is intended.

We should perhaps check that the notation of Euler's formula agrees with what we know about the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ by $\exp(x) = e^x$ defined for real values of x. For example, if $\theta = 0$, then our notational convention gives

$$e^{i\ 0} = \cos(0) + i\sin(0) = 1 + 0\ i = 1.$$

This agrees with the fact that $e^0 = 1$. This is the only way to get a real power of e in the formula $e^{i\theta}$ by taking $\theta \in \mathbb{R}$. The value of e^w where w is a complex number is not introduced in BC until Chapter 3. This is a little bit cumbersome, but for now, we should remember that e^w where w is a complex number is only defined along the imaginary axis where $w = i\theta$ for some real number θ.

In Chapter 3 of BC one finds the definition

$$e^w = e^{\text{Re}(w)} e^{i\,\text{Im}(w)} = e^{\text{Re}(w)}[\cos \text{Im}(w) + i \sin \text{Im}(w)].$$

For comparison, you may think about how the real exponential function is defined in calculus. There are several options that make pretty good sense. One reasonable approach is to use a power series:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n. \tag{1.29}$$

Another possibility, if one knows about existence and uniqueness for ODEs, is to define $\exp : \mathbb{R} \to \mathbb{R}$ to be the unique solution of the initial value problem

$$\begin{cases} y' = y, & x \in \mathbb{R} \\ y(0) = 1. \end{cases} \tag{1.30}$$

One must admit that a lot of work is involved in showing these two solid definitions lead to the same function $\exp(x) = e^x$ and then (on top of that) showing the resulting function has all the nice properties one expects of an exponential.

Exercise 1.41 Show the power series solution of the initial value problem (1.30) is given by (1.29).

Exercise 1.42 Use one of the definitions of the real exponential function above to show

$$e^{a+b} = e^a \ e^b \qquad \text{for} \qquad a, b \in \mathbb{R}.$$

The power series approach for defining the real exponential mentioned above can be generalized to give a (rigorous) definition of the complex exponential function. This requires a discussion of (complex) power series (Chapter 5 in BC which will sort of be at the end of our course). In fact, the differential equations approach can be generalized as well, but for that one needs to know something (at the very least) about complex derivatives. We'll be coming to that relatively soon (Chapter 2 in BC). For now, we should just be careful that we only use e^w when $w \in i\mathbb{R} \cup \mathbb{R}$.

We can consider, for example, the expressions

$$e^{i(a+b)} \qquad \text{and} \qquad e^{ia}e^{ib}$$

where $a, b \in \mathbb{R}$. The first expression is

$$e^{i(a+b)} = \cos(a+b) + i\sin(a+b) = \cos a \cos b - \sin a \sin b + (\sin a \cos b + \sin b \cos a)i.$$

The second expression is

$$e^{ia}e^{ib} = (\cos a + i \sin a)(\cos b + i \sin b) = \cos a \cos b - \sin a \sin b + (\cos a \sin b + \sin a \cos b)i.$$

Comparing the two, we obtain the "formula"

$$e^{i(a+b)} = e^{ia}e^{ib}.$$

Exercise 1.43 Verify **de Moivre's formula**:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Exercise 1.44 Verify the following for $a, b, \theta, \psi \in \mathbb{R}$:

(a) $(ae^{i\theta})(be^{i\phi}) = abe^{i(\theta+\phi)}$.

(b) $e^{i\theta} \neq 0$ (ever!).

(c) If $b \neq 0$, then

$$\frac{ae^{i\theta}}{be^{i\phi}} = \frac{a}{b}e^{i(\theta-\phi)}.$$

(d) $(ae^{i\theta})^n = a^n e^{in\theta}$ for $n \in \mathbb{Z} = \{0, \pm1, \pm2, \pm3, \ldots\}$.

Section 1.9 in BC concerns the geometric iterpretation of the complex product. We have essentially already covered this, but there is one thing to note: It is, of course, the case that

$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)},$$

so it makes sense to write

$$\arg(e^{i\theta}e^{i\phi}) = \arg e^{i(\theta+\phi)},$$

but it does not necessarily follow from this that if

$$e^{i\theta}e^{i\phi} = e^{i\psi},$$

then $\psi = \theta + \phi$, nor does it make sense to write $\phi = \psi - \theta$. What one can actually write is this:

$$\psi = \theta + \phi + 2\pi k \qquad \text{for some } k \in \mathbb{Z} = \{0, \pm1, \pm2, \pm3, \ldots\}.$$

Similarly, we can say

$$\phi = \psi - \theta + 2\pi\ell \qquad \text{for some } \ell \in \mathbb{Z} = \{0, \pm1, \pm2, \pm3, \ldots\}.$$

Here is a computation inspired by Exercise 5 part (a) in BC:

$$\begin{aligned} i(1 - i\sqrt{3})(\sqrt{3} + i) &= e^{i\pi/2}\ (2)\ e^{-i\pi/3}\ (2)\ e^{i\pi/6} \\ &= 4e^{i\pi(1/2-1/3+1/6)} \\ &= 4e^{i\pi/3} \\ &= 4\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ &= 2(1 + i\sqrt{3}). \end{aligned}$$

In the first step, when I look at a complex number like $1-i\sqrt{3}$, I compute it's modulus and find the value 2. This tells me $1-i\sqrt{3}$ is on the circle or radius 2 centered at the origin, and its exponential form is $2e^{i\theta}$. Then I picture the **quadrant** of the complex plane, or more precisely, the quadrant of the unit circle in which $(1/2 - i\sqrt{3}/2) = e^{i\theta}$ must be located. In this case, the number is in the fourth quadrant. Based on the fact that the abscissa (real part) is smaller than the (absolute value of the) ordinate (imaginary part) and the fact that I recognize these numbers, I can determine the/an argument. In short, I recognize

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \qquad \text{and} \qquad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

(It helps, of course, to be familiar with the values of the trigonometric functions of certain often used-angles like $\pi/6, \pi/4, \pi/3$ and $\pi/2$. The more you know, the more you know!)

Applying the same process to the other factor, I found (of course)

$$\sqrt{3} + i = 2e^{i\pi/6}.$$

Let's cube this number:

$$(\sqrt{3} + i)^3 = 8e^{i\pi/2} = 8i.$$

So that's sort of interesting. If we take the same number to the sixth power, we get

$$(\sqrt{3} + i)^6 = 2^6(-1) = -64,$$

and if we go all the way to the twelfth power

$$(\sqrt{3} + i)^{12} = 2^{12}e^{i(2\pi)} = 2^{12} \qquad \big(= 4096\big).$$

Let's start with each of these numbers $8i$, $-64 = 2^6$ and 2^{12} and attempt to reverse taking the power, that is, let's try to take the corresponding root. In the last case, the situation is quite interesting, because a very different twelfth root of 2^{12} is obvious, namely 2. That is,

$$2^{12} = 2^{12} \qquad \text{and} \qquad (\sqrt{3} + i)^{12} = 2^{12}.$$

This might not be surprising if you know the complex dodecic polynomial equation $z^{12} - 2^{12} = 0$ is (at least likely to) have twelve roots. You might imagine at first, the root 2 is a repeated root, repeated twelve times, but then you might remember how to factor

$$z^{12}-2^{12} = (z-2)(z^{11}+2z^{10}+2^2z^9+2^3z^8+2^4z^7+2^5z^6+2^6z^5+2^7z^4+2^8z^3+2^9z^2+2^{10}z+2^{11}),$$

and 2 is not (even close to) a root of the hendecic polynomial equation

$$z^{11} + 2z^{10} + 2^2z^9 + 2^3z^8 + 2^4z^7 + 2^5z^6 + 2^6z^5 + 2^7z^4 + 2^8z^3 + 2^9z^2 + 2^{10}z + 2^{11} = 0,$$

so if we believe the fundamental theorem of algebra, there ought to be at least one more root out there. And we know there is: $\sqrt{3} + i$. To find the roots, let's look for complex numbers in exponential form $z = re^{i\theta}$ with

$$r^{12}e^{12i\theta} = 2^{12} = 2^{12}e^{2\pi i}.$$

Taking the modulus of each side we must have $r^{12} = 2^{12}$. We have our trusty root $r = 2$. This is really the only one we need and, it turns out, the only positive one we can get. We can see this by noting that when $z \geq 0$

$$z^{11} + 2z^{10} + 2^2z^9 + 2^3z^8 + 2^4z^7 + 2^5z^6 + 2^6z^5 + 2^7z^4 + 2^8z^3 + 2^9z^2 + 2^{10}z + 2^{11} \geq 2^{11}.$$

We can also do some more factoring:

$$\begin{aligned} z^{12} - 2^{12} &= (z^6 - 2^6)(z^6 + 2^6) \\ &= (z^2 - 2^2)(z^4 + 2^2z^2 + 2^4)(z^6 + 2^6) \\ &= (z - 2)(z + 2)(z^4 + 2^2z^2 + 2^4)(z^6 + 2^6). \end{aligned}$$

In this way, we see there is precisely one negative real root $z = -2$ which is also a (third) twelfth root of 2^{12}, which we should have known was there from the beginning.

In summary, we have found three twelfth roots of 2^{12}, namely, ± 2 and $\sqrt{3}+i$. We should recall that, by definition, the polar form of a complex number $z = re^{i\theta}$ has $r = |z| \geq 0$, so there is really only one possibility for r, and that is $r = 2$. Returning to the search for $z = re^{i\theta}$, we consider

$$e^{12i\theta} = e^{2\pi i}.$$

This means, of course, $12\theta = 2\pi + 2\pi k = 2\pi(1 + k)$ for some $k \in \mathbb{Z}$. If we take $k = 0$, we get $\theta = \pi/6$ which gives $z = \sqrt{3} + i$. If we take $k = 1$, we get a new root:

$$\theta = \frac{\pi}{6}(k + 1) = \frac{\pi}{3} \qquad \text{and} \qquad z = 2e^{i\pi/3} = 1 + i\sqrt{3}.$$

In fact, $(1+i\sqrt{3})^3 = 8e^{i\pi} = -8$, so this is a cube root of -8 and $(-8)^4 = 2^{12}$.

We have found four distinct complex twelfth roots of 2^{12}, and we can see all the rest. They correspond to

$$\theta = \ell\frac{\pi}{6} \qquad \text{for} \qquad \ell \in \mathbb{Z}.$$

The next one is... $z = 2i$. Then you just go around the circle:

$$\begin{aligned}
2e^{2\pi/3} &= -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \\
2e^{5\pi/6} &= -\frac{\sqrt{3}}{2} + \frac{i}{2}, \\
2e^{-\pi} &= -2, \qquad \text{(which we already found)} \\
2e^{7\pi/6} &= -\frac{\sqrt{3}}{2} - \frac{i}{2}, \\
2e^{4\pi/3} &= -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \\
2e^{2\pi/3} &= -2i, \\
2e^{5\pi/3} &= \frac{1}{2} - i\frac{\sqrt{3}}{2}, \\
2e^{11\pi/6} &= \frac{\sqrt{3}}{2} - \frac{i}{2}.
\end{aligned}$$

It may be noted that the last root $z = 2e^{11\pi/6}$ corresponds to $\ell = 11$ and $k = 10$ in

$$z = 2e^{\ell i\pi/6} = 2e^{(k+1)i\pi/6}.$$

After this, we can take $\ell = k+1 = 12, 13, 14, 15, \ldots$, and the list will just repeat starting with $z = 2$. Similarly, taking $\ell = k+1 = 0$ we get $z = 2$ again, and taking $\ell = k+1 = -1, -2, -3, \ldots$ we get the same roots in reverse order. Conclusion: We have found twelve distinct complex roots of 2^{12} and these are all the complex roots of 2^{12}. They are symmetrically spaced around the circle $\partial B_2(0) \subset \mathbb{C}$ as indicated in Figure 1.4

Exercise 1.45 Find the six complex sixth roots of -64.

Exercise 1.46 Find the three complex third roots of $8i$.

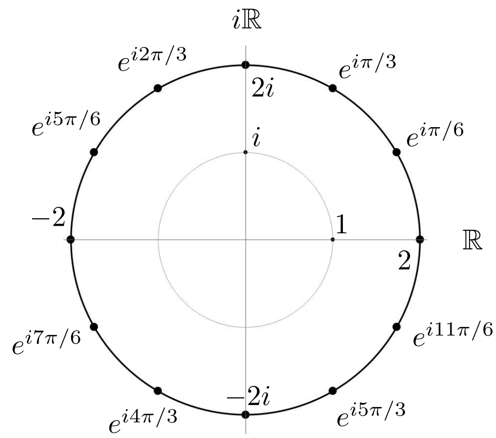

Figure 1.4: The twelve complex twelfth roots of 4096.

It is more or less clear that the approach to finding roots above will work for any complex number and any positive integer: Write $w = |w|e^{i\theta_0}$ for $w \in \mathbb{C}\backslash\{0\}$ where $\theta_0 = \arg(w)$ and $z = re^{i\theta}$. For any $n \in \mathbb{N}$, there is exactly one positive real root $r = \sqrt[n]{|w|}$ of $r^n = |w|$. Then take

$$\theta = \frac{1}{n}(\theta_0 + 2\pi k) = \frac{\theta_0}{n} + \frac{2\pi k}{n} \qquad \text{for} \qquad k \in \mathbb{Z}.$$

The distinct complex n-th roots of w are thus given by

$$\sqrt[n]{|w|}\; e^{i(\theta_0+2\pi k)/n} = \sqrt[n]{|w|}\; e^{i\theta_0/n}\; e^{i2\pi k/n} \qquad \text{for} \qquad k = 0, 1, 2, \ldots, n-1.$$

Exercise 1.47 Given $n \in \mathbb{N}$, the numbers

$$\zeta_n = e^{i2\pi k/n} \qquad \text{for} \qquad k = 0, 1, 2, \ldots, n-1$$

are called the **n-th roots of unity**. Show the set of complex roots of the polynomial equation $z^n - 1 = 0$ is precisely

$$\{1, \zeta_1, \zeta_2, \ldots, \zeta_{n-1}\}.$$

What is the relation between the n-th roots of unity and the n-th roots of -1, i.e., the solutions

$$\{i, \omega_1, \omega_2, \ldots, \omega_{n-1}\}$$

of the polynomial equation $z^n + 1 = 0$?

1.7 A little topology in $\mathbb{C}$

We have already defined the "open" ball

$$B_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

determined by a complex number z_0 and a positive radius r. Such a set is also sometimes called a **neighborhood**. Brown and Churchill introduce this notion in their section 12. More generally, any set $\mathcal{N} \subset \mathbb{C}$ with $z_0 \in \mathcal{N}$ is called a **neighborhood** of z_0 if there is a ball $B_r(z_0)$ with

$$B_r(z_0) \subset \mathcal{N}.$$

Here is an important definition:

Definition 1 *A set $U \subset \mathbb{C}$ is* **open** *if for each point $z_0 \in U$ there is some open ball $B_r(z_0)$ with*

$$B_r(z_0) \subset U.$$

Exercise 1.48 Explain why not every neighborhood of a point $z_0 \in \mathbb{C}$ is open in $\mathbb{C}$. Explain also why every open set in $\mathbb{C}$ is a neighborhood of each of its points. Finally, show that an open ball $B_r(z_0)$ in $\mathbb{C}$ is open in $\mathbb{C}$.

Interior points an the interior of a set

Given a point z is a set $A \subset \mathbb{C}$, we say z is an **interior point** of A if there is some open ball $B_r(z)$ with $B_r(z) \subset A$. Note this condition on points plays the star role in the definition of an open set: A set is open if each of its points is an interior point. A general set, however, can have some points that are interior points and some points that are not interior points. Consider for example $A_1 = B_1(0) \cup \{1\}$ or $A_2 = B_1(0) \cup \{\pm 1, \pm i\}$ or $A_3 = B_1(0) \cup \{\pm 1, \pm i, 2\}$.

The **interior of a set** $A \subset \mathbb{C}$ is the collection of all interior points, and we can write

$$\text{int}(A) = \{z \in A : \text{there exists some } r > 0 \text{ with } B_r(z) \subset A\}.$$

Every subset of $\mathbb{C}$ has an interior, but the interior may be empty.

A point $z \in \mathbb{C}$ is **exterior to** A, or is an **exterior point** of A if z is in the interior of the complement of A. Here is some notation for that:

$$A^c = \{z \in \mathbb{C} : z \notin A\} \qquad \text{and} \qquad \text{ext}(A) = \text{int}(A^c).$$

As the notation suggests: The **exterior of a set** $A \subset \mathbb{C}$ is the collection of all points in $\mathbb{C}$ exterior to A.

Exercise 1.49 Find the interior and exterior of the sets:

(a) $A_1 = B_1(0) \cup \{1\}$,

(b) $A_2 = B_1(0) \cup \{\pm 1, \pm i\}$, and

(c) $A_3 = B_1(0) \cup \{\pm 1, \pm i, 2\}$.

Show that in general, given any set $A \subset \mathbb{C}$, the sets $\mathrm{int}(A)$ and $\mathrm{ext}(A)$ are open sets.

Boundary points and the closure

In general, not every point in $\mathbb{C}$ is either exterior to or interior to a given set A. Some points satisfy neither condition. But the complement of the interior and exterior is called the **boundary**:

$$\mathbb{C} = \mathrm{int}(A) \cup \mathrm{ext}(A) \cup \partial A. \tag{1.31}$$

This is a very strange way to introduce the topological boundary of a set, and I apologize (a little bit) for that. Let me give the boundary another (more standard) introduction.

A point $z \in \mathbb{C}$ is a **boundary point of a set** $A \subset \mathbb{C}$ if every open ball $B_r(z)$ intersects both A and A^c, that is, given $r > 0$

$$A \cap B_r(z) \neq \phi \qquad \text{and} \qquad A^c \cap B_r(z) \neq \phi.$$

The **boundary of a set** $A \subset \mathbb{C}$ is the collection of all boundary points and is denoted by ∂A. Now you can prove (1.31) from which is follows that no matter what set $A \subset \mathbb{C}$ you may encounter at least one of the sets $\mathrm{int}(A)$, $\mathrm{ext}(A)$, or ∂A is nonempty.

Exercise 1.50 What can you say about a set $A \subset \mathbb{C}$ if

(a) $\mathrm{int}(A) = \phi$,

(b) $\mathrm{ext}(A) = \phi$, or

(c) $\partial A = \phi$?

Here is another (surprisingly unsymmetric) important definition:

Definition 2 (closed set) A set $A \subset \mathbb{C}$ is **closed** if A^c is open.

There are lots of questions to ask (and answer) now. Here are a few to get you started:

Exercise 1.51 Show $\text{int}(A) = A \backslash \partial A$.

Exercise 1.52 Show **any** union of open sets is open. More precisely, if $\{U_\alpha\}_{\alpha \in \Gamma}$ is a collection of open sets (in $\mathbb{C}$) where Γ is an **arbitrary indexing set**, i.e., Γ could be for example $\mathbb{N}$, $\mathbb{R}$ or $\mathbb{C}$, then

$$\bigcup_{\alpha \in \Gamma} U_\alpha = \{z \in \mathbb{C} : \text{there is some } \alpha \in \Gamma \text{ with } z \in U_\alpha\} \quad \text{is open.}$$

Exercise 1.53 Show any finite intersection of open sets is open. That is, given $U_1, U_2, \ldots, U_n$ open sets in $\mathbb{C}$, the intersection

$$\bigcap_{j=1}^{n} U_j \quad \text{is open.}$$

(Here the indexing set on the union is $\{1, 2, \ldots, n\}$.)

Exercise 1.54 Show finite unions and arbitrary intersections of closed sets are closed.

Exercise 1.55 Show the following are equivalent conditions on a set $A \subset \mathbb{C}$:

(i) A is open.

(ii) $A = \text{int}(A)$.

(iii) $A \cap \partial A = \phi$.

Exercise 1.56 Show a set $A \subset \mathbb{C}$ is closed if and only if $\partial A \subset A$.

Exercise 1.57 Show some sets are neither open nor closed and some sets are both open and closed. Show ∂A is always closed.

The **closure**[8] of any set $A \subset \mathbb{C}$ is given by

$$\overline{A} = A \ \cup \ (\partial A).$$

Exercise 1.58 Show a set $A \subset \mathbb{C}$ is closed if and only if $\overline{A} = A$.

[8] The notation for the closure of a set looks like the notation for the conjugation of a complex number, but the two have little or nothing to do with each other.

Punctured neighborhoods and accumulation points

A neighborhood of $z \in \mathbb{C}$, and especially an open ball $B_r(z)$ is said to be **punctured** at z if z is excluded from the neighborhood. This is a little tricky, or more subtle than it sounds, because a "punctured neighborhood" is (technically) not a neighborhood.

A punctured ball, or what is much more usually referred to as the **punctured disk**, with center $z \in \mathbb{C}$ is

$$B_r(z)\backslash\{z\}.$$

A point $z \in \mathbb{C}$ is said to be an **accumulation point** of the set $A \subset \mathbb{C}$ if every punctured disk $B_r(z)\backslash\{z\}$ intersects A:

$$A \cap (B_r(z)\backslash\{z\}) \neq \phi.$$

The set of all accumulation points of a set A is denoted by $\text{acc}(A)$.

Exercise 1.59 Find the accumulation points of the sets from Exercise 1.49:

(a) $A_1 = B_1(0) \cup \{1\}$,

(b) $A_2 = B_1(0) \cup \{\pm 1, \pm i\}$, and

(c) $A_3 = B_1(0) \cup \{\pm 1, \pm i, 2\}$.

Show that in general, the set of accumulation points $\text{acc}(A)$ can be open, closed, or neither.

Exercise 1.60 (Exercises 1.12.8-10 in BC) Show the following:

(a) $A \subset \mathbb{C}$ is closed if and only if $\text{acc}(A) \subset A$.

(b) If $A \subset \mathbb{C}$ is open then $A \subset \text{acc}(A)$.

(c) If $\text{acc}(A) \neq \phi$, then A must have infinitely many (distinct) points.

Intermission

So then we have a set of complex numbers $\mathbb{C}$. It has algebraic and analytic structure which can be compared to familar sets like $\mathbb{R}$ and $\mathbb{R}^2$. In particular, it is an algebraically complete field. We have spent a good deal of time addressing some of the basic properties of $\mathbb{C}$ partially because Brown and Churchill have done so. It is, at some level, time to leave such matters to rest for a while and move on to chapter 2 and a discussion of functions $f : U \to \mathbb{C}$ where U is an open subset of $\mathbb{C}$ and a discussion of the **differentiability** of such a function in particular. I offer a kind of quick review or perspective before we begin.

I may have mentioned that $\mathbb{R}$, aside from being an algebraic field, is analyticaly an **ordered Archimedean field with the least upper bound property**. It fact, I believe it's a theorem in real analysis somewhere that, up to some kind of isomorphism, $\mathbb{R}$ is the only such set. In comparison, $\mathbb{C}$ is not ordered. Archimedean means that for each element a, there is another b with $a < b$. Since there is no (natural) order on $\mathbb{C}$, the Archimedean property is not applicable to $\mathbb{C}$ either. Similarly, the property that each set which is bounded above has a least upper bound can only apply to ordered sets.

There are words that describe the analytic structure of $\mathbb{C}$. I may have mentioned that $\mathbb{C}$ is a **complete Hermitian inner product field**. These words are, in some sense, not so interesting for us because the point of the course is to understand the field $\mathbb{C}$ as a kind of prototypical example of a set with these analytic properties. I will, however, make a couple comments (below).

I may have also mentioned that the fact that $\mathbb{R}$ is an **ordered Archimedean field with the least upper bound property** is fundamentally what leads to the subject of **calculus,** a.k.a. real analysis, where one studies differentiation and integration of real valued functions of a real variable. And finally, I may have mentioned the fact that when one attempts to discuss differentiation and integration for complex functions of a complex variable, the subject is what is known (at least to the old Europeans—especially Germans—in the 1700s and 1800s) as **analytic function theory** and this subject has a character involving certain complications—in short it is complex. The complexity is primarily owing to the structure of the **complex product**

$$(a+bi)(x+iy) = ax - by + (ay+bx)i \qquad \text{(the complex product)}$$

and its geometric relation to **rotation and scaling** mentioned above or more simply due to the curious relation

$$i^2 = -1.$$

In this regard, we have decomposed a complex number in a second way (in addition to decomposition into real and imaginary parts) into the "modulus part" and the "argument part," and introduced the notation of Euler:

$$z = |z|e^{i\arg(z)}.$$

I mentioned that the product is a little cumbersome at first which may be construed to mean it is a little unnatural for (most or many) humans. You should of course not be discouraged by this. Think of all the time it took (you) to learn to add and multiply (and subtract and divide) "natural numbers." You still don't even know all the primes. But lots of relatively complicated mathematical manipulations are now second nature to you. These can all be thought of as somewhat unnatural things for humans to think about. There was probably a time in the past when most humans focused more time on useful activities like swimming and staying healthy, hunting and making fire without matches. I guess the activity of multiplying should also be included in this list but not in the mathematical sense. Currently however the fashion is for most humans to spend a lot of time sitting around learning to add and multiply in the mathematical sense, burning lots of fossil fuels, and doing other equally meaningful activities like playing computer games and studying complex analysis like the old Germans.

There is also an **inner product** on $\mathbb{C}$. It is different from the product:

$$\langle a+bi, x+iy\rangle = (a+bi)(x-iy) = ax+by+(bx-ay)i. \qquad \text{(the complex inner product)}$$

You know this Hermitian inner product mostly in the form of the associated **norm** called the **complex absolute value**

$$|a+bi| = \sqrt{\langle a+bi, a+bi\rangle} = \sqrt{a^2+b^2}$$

and the associated **metric distance**

$$d(z,w) = |z-w|$$

which makes $\mathbb{C}$ a **metric space**. Incidentally, it has been mentioned that $\mathbb{C}$ is algebraically complete, but "complete" in the analytic description "complete Hermitian inner product field," is something different. This is analytically complete or **metrically complete**, meaning every Cauchy sequence has a limit in the (metric) space. This is a property also shared by the ordered Archimedean field with the least upper bound property (known commonly as $\mathbb{R}$).

If you're unfamiliar with metric spaces and Cauchy sequences and such things, do not worry. You learned calculus without knowing important things involving those concepts about $\mathbb{R}$, and you can learn some analytic function theory without knowing lots of tedious details about $\mathbb{C}$. Nobody knows everything about $\mathbb{N} = \{1, 2, 3, \ldots\}$ or $\mathbb{R}$ or $\mathbb{C}$.

Chapter 2

A Complex Variable

If we're going to talk about a function $f : U \to \mathbb{C}$ where U is some (probably open) subset of $\mathbb{C}$, then we need to be familiar with (and comfortable with) the basics of functions in general.

Generally given two sets X and Y, a **function** is a rule or correspondence which assigns to each $x \in X$ a unique $y \in Y$.

Note carefully, the use of the words **"to each"** and **"a unique"** in the definition of a function and make sure you understand their meaning thoroughly. Given a function $f : X \to Y$ and an element $x \in X$, the unique element assigned to x is denoted by $f(x)$.

Along with the definition[1] of function, there are various other definitions it can be helpful to keep in mind:

Definition 3 Given a function $f : X \to Y$, the set X is called the **domain** of the function f and the set Y is (at least sometimes) called the **codomain** of the function f. The **range** of f is the set

$$\{f(x) : x \in X\}.$$

The range is also sometimes called the **image**, or the **image of X under f and is denoted by** $f(X)$:

$$f(X) = \{f(x) : x \in X\}.$$

[1]The definition we are using here is usually attributed to Leonhard Euler who had the basic idea around 1750, though the precise formulation we have used involving arbitrary sets was probably only first written down by Dedikind around 1880. Up until that time, most people probably thought of functions as mostly real valued functions of a real variable or even continuous real valued functions of a real variable. It appears that Lobachevsky and Dirichlet were getting pretty close in the 1830s.

More generally, the **image of any subset** A **of the domain** is

$$f(A) = \{f(x) : x \in A\}.$$

The **preimage** of a set $S \subset Y$ is

$$f^{-1}(S) = \{x \in X : f(x) \in S\}.$$

Note the preimage of a set $S \subset Y$ is a subset of the domain X. Note also that the notation $f^{-1}(S)$ is not intended to mean, and does **not** mean in general, that there exists a function $g : Y \to X$ for which

$$g \circ f(x) = x. \tag{2.1}$$

Definition 4 A function $f : X \to Y$ is **one-to-one** if for each $y \in f(X)$, there exists a unique $x \in X$ such that $f(x) = y$. This condition on a function is also often expressed by saying the function is **1-to-1**, **1-1**, or **injective**.

Exercise 2.1 Show $f : X \to Y$ is injective if and only if

$$f^{-1}\{f(x)\} \quad \text{is a singleton for each } x \in X.$$

Definition 5 A function $f : X \to Y$ is said to be **surjective** or **onto** if for each $y \in Y$, there exists some $x \in X$ with $f(x) = y$.

Definition 6 If a function $f : X \to Y$ is both surjective and injective, or in other words one-to-one and onto, then the function is said to be **bijective** or a **one-to-one correspondence**. It is in this circumstance that there exists a function $g : Y \to X$ for which (2.1) holds.

Incidentally, the notation $g \circ f$ appearing in (2.1) is a special case of the following: If $f : X \to Y$ and $g : Y \to Z$ are functions, then there is a function $g \circ f : X \to Z$ called the **composition** of g on f with values

$$g \circ f(x) = g(f(x)).$$

Exercise 2.2 Show $f : X \to Y$ is bijective if and only if there exists a function $g : Y \to X$ for which

$$g \circ f(x) = x \text{ for all } x \in X \qquad \text{and} \qquad f \circ g(y) = y \text{ for all } y \in Y. \tag{2.2}$$

In this case we write $g = f^{-1} : Y \to X$, and we can write $f^{-1}(y)$ for elements $y \in Y$ instead of only being able to write $f^{-1}(S)$ for sets $S \subset Y$. That is, in this case, f^{-1} denotes a function and is not just a notation used to denote the inverse images of sets. If there is such a function $f^{-1} : Y \to X$, then f is said to be **invertible** and f^{-1} is called the **inverse** of f.

The preimage of a set $f^{-1}(S)$ is also sometimes called the inverse image of the set, though again it should always be remembered that seeing someone freely tossing around the inverse images of sets (or you doing it yourself) should not be interpreted to mean there is an inverse function f^{-1} floating around somewhere.

Definition 7 Given a function $f : X \to Y$ and a set $A \subset X$, there is a (different) function

$$f\big|_A : A \to Y$$

with values given by

$$f\big|_A(x) = f(x) \qquad \text{for } x \in A.$$

This function is called the **restriction** of f to the set A.

Hopefully, that is an adequate collection of important words and concepts for our impending discussion of complex functions of a complex variable $f : U \to \mathbb{C}$. Certainly all these concepts are going to be used/needed/important.

One definition that is good to know in general but which is not so important or commonly used in complex analysis is the following:

Definition 8 The **graph** of a function $f : X \to Y$ is the set

$$\{(x, f(x)) \in X \times Y : x \in X\}.$$

Incidentally, the set of all ordered pairs (x, y) for which $x \in X$ and $y \in Y$ is called the **Cartesian product** of the sets X and Y. The graph of a function is a subset of the Cartesian product of the domain and codomain.

Another good thing to know (or realize or keep in mind) in general, though almost not worth mentioning, is that given any set X there is always a function, called the **identity** function $\mathrm{id} : X \to X$ with values $\mathrm{id}(x) = x$ for all $x \in X$. The identity function is always bijective and is its own inverse.

Exercise 2.3 Use the identity function(s) to simplify the inverse relations (2.2).

2.1 Some functions $f : U \to \mathbb{C}$ with $U \subset \mathbb{C}$

Many functions come directly from the complex arithmetic considered in Chapter 1 above. For example, we can consider $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ by

$$f(z) = \frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}\, i$$

where $\text{Re}(z) = x$ and $\text{Im}(z) = y$ as usual. Notice the domain of this function is $U = \mathbb{C}\backslash\{0\}$ which is an open subset of $\mathbb{C}$.

The **real and imaginary parts of the function value** $f(z)$ play an important part in the discussion of complex valued functions of a complex variable. As perhaps suggested by the expression for $1/z$ above, we usually consider $u = \text{Re}(f)$ and $v = \text{Im}(f)$ as **real valued functions of two real variables** like functions considered in elementary calculus (III). In the case of the reciprocal function, $u : \gamma^{-1}(U) \to \mathbb{R}$ by

$$u(x,y) = \frac{x}{x^2+y^2}.$$

Similarly, $v : \gamma^{-1}(U) \to \mathbb{R}$ by

$$v(x,y) = -\frac{y}{x^2+y^2}.$$

Remember $\gamma : \mathbb{R}^2 \to \mathbb{C}$ the isomorphism giving $\gamma^{-1}(\mathbb{C}\backslash\{0\}) = \mathbb{R}^2\backslash\{(0,0)\}$.

Exercise 2.4 Draw the graphs of $u = \text{Re}(f)$ and $v = \text{Im}(f)$ when $f(z) = 1/z$.

Incidentally, we could think of the real and imaginary parts of a function $f : U \to \mathbb{C}$ where $U \subset \mathbb{C}$ as real valued functions of a complex variable. If we did that, the values of $\text{Re}(1/z)$ would look something like

$$\frac{\text{Re}(z)}{|z|^2}.$$

The reciprocal function is perhaps a complicated function to start with. Shall we consider something a bit easier? Take for example, the identity function

$$\text{id} : \mathbb{C} \to \mathbb{C} \qquad \text{by } \text{id}(z) = z.$$

You might not think there is much to say about this complex function, and maybe you are sort of right. Let me try to say something anyway. First of all, the graph of the identity function, which happens to be the **diagonal**

$$\{(z,z) \in \mathbb{C}^2 : z \in \mathbb{C}\}$$

in $\mathbb{C}^2$ is already difficult to "see" (or draw).[2]

Exercise 2.5 What can you say about

$$\{(\gamma^{-1}(z), \gamma^{-1}(z)) = (x, y, x, y) \in \mathbb{R}^4 : z = x + iy \in \mathbb{C}\}$$

as a subset of $\mathbb{R}^4$?

Still, there is a pretty nice geometric picture associated with (this and many other) complex functions. Roughly, we can think of the domain $\mathbb{C}$ of id as "taken" by the identity function and "laid down" on a **second copy**, a codomain copy, of $\mathbb{C}$. What I am trying to express is quite easy—it's so easy for the identity function, you might have difficulty seeing it. If so, do not worry. You can come back to what I've just said in a moment, and it should make perfect sense. Just remember that what we are talking about here is what is called (by some people at least) a **mapping picture**, and using a mapping picture is an alternative way to visualize the "workings" of a function, especially a complex valued function—an alternative to trying to draw (or imagine) the graph of the function.

To add additional foreshadowing, I might mention that $\text{id}(z) = z$ has a **complex derivative** and you might not be surprised to be informed that that derivative is the **constant function 1** : $\mathbb{C} \to \mathbb{C}$ by $\mathbf{1}(z) \equiv 1$. Can you give the mapping picture for **1**? What I would guess you would not know is the definition, meaning, or how to compute a complex derivative. Those are among the main subjects of this chapter, so for now you can look forward to (soon) really understanding what it means that

$$\text{id}'(z) \equiv 1.$$

A good place to really start with mapping pictures is with the complex square function (with) $f(z) = z^2$. On the one hand,

$$z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi,$$

So the real and imaginary parts of f are given by the familiar functions[3]

$$u(x, y) = x^2 - y^2 \qquad \text{and} \qquad v(x, y) = 2xy.$$

[2] I may have mentioned that the subject of several complex variables is a notoriously difficult subject. You can start to see that even here.

[3] The names u and v for the real and imaginary parts of a complex valued function of a complex variable $z = x + iy$ are almost as standard as naming a function f or the designations/conventions $x = \text{Re}(z)$ and $y = \text{Im}(z)$.

On the other hand, squaring is a kind of multiplication, and we know multiplication has its primary geometric interpretation through scaling and rotation:

$$z^2 = |z|^2 e^{2i\theta} \qquad \text{where} \qquad \theta = \arg(z).$$

Notice the rotation in this case corresponds to doubling the argument. In particular, for $z \in \mathbb{S}^1$ (the unit circle) where $|z| = 1$, the doubling of the angle essentially tells the whole story. For example, we can say the semicircle $\mathbb{S}^+ = \{e^{i\theta} \in \mathbb{C} : 0 \leq \theta < \pi\}$ is mapped bijectively and monotonically onto the entire unit circle. In more colorful language, the semicircle $\mathbb{S}^+$ is **stretched** around the entire unit circle starting from $1^2 = 1$ with a uniform doubling of **length** so as to **cover** all of $\mathbb{S}^1$. See Figure 2.1.

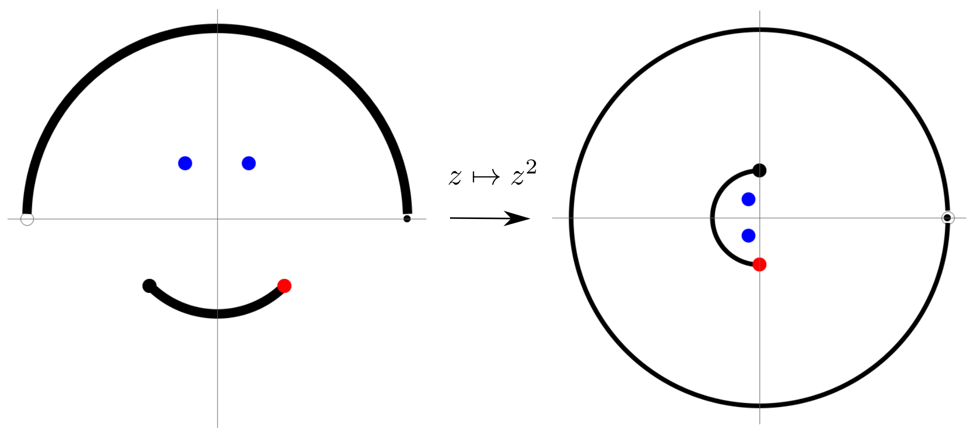

Figure 2.1: A mapping picture for $f(z) = z^2$. Here we show the images of certain arcs of circles in $\mathbb{C}$. In particular, the image $\mathbb{S}^1$ of the semicircle $\mathbb{S}^{+1}$, known as the **upper unit semicircle** in $\mathbb{C}$, is shown on the right. The width of the image circle in this graphic is half of that used to plot $\mathbb{S}^+$ on the left (to indicate the stretching of length by a factor of 2 along the circle. Can you determine which blue point in the domain goes to which blue point in the image? .

The images of certain other points and sets are indicated in Figure 2.1. Let's take a look at those. First make sure you understand what is happening with the upper unit semicircle $\mathbb{S}^+$. If we were to extend the mapping to $e^{i\theta}$ with $\theta = \pi$, then we would get image $(e^{i\pi})^2 = 1$ which would overlap the image of $\mathbb{S}^+$ and then we would start to cover the circle in the image a second time. This suggests consideration of a kind of **second copy** of $\mathbb{C}$ to keep track of what's happening in the codomain. I have

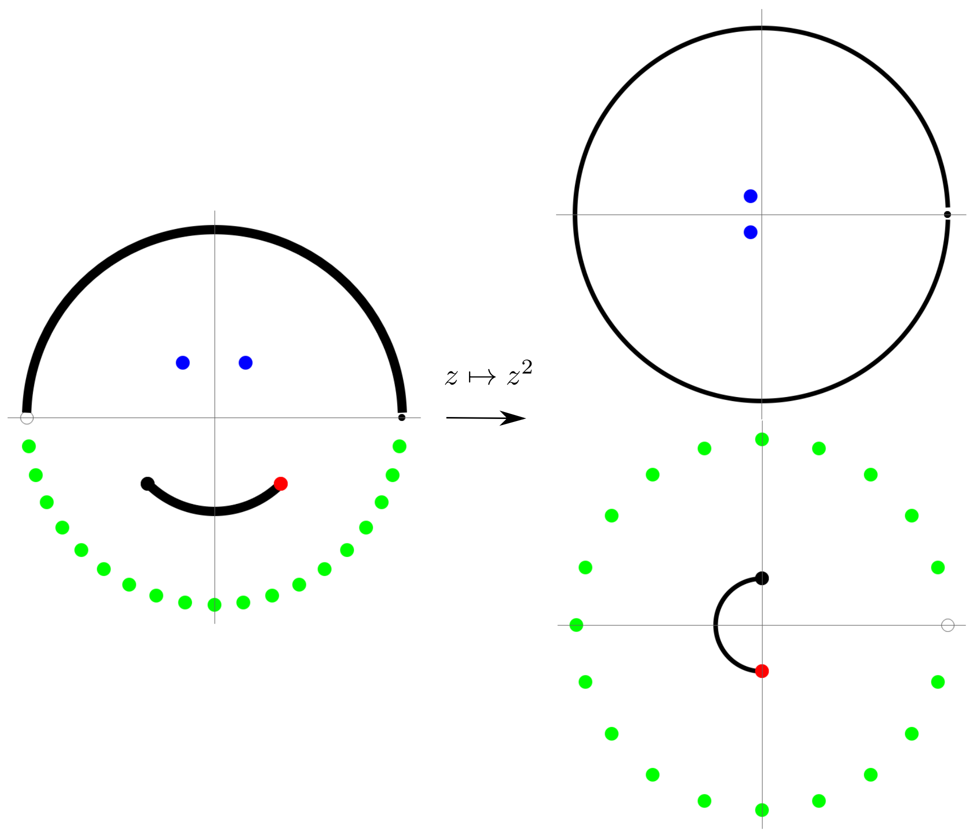

Figure 2.2: The codomain for $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$ is nominally $\mathbb{C}$, but it makes for better bookkeeping if we separate the codomain into two "sheets." These two sheets, separated and properly glued together along a "branch cut" are the domain for the complex square root function. .

drawn a mapping picture using this idea in Figure 2.2. The green points along the lower unit semicircle get stretched, starting with $e^{i\pi} = -1 \mapsto e^{2\pi i} = 1$, around the entire unit circle $\mathbb{S}^1$ a second time. In order to keep track of these images, we have introduced a second "sheet" of $\mathbb{C}$ in the codomain. The restriction

$$f\Big|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{S}^1$$

does not have an inverse because, though this restriction is surjective onto the unit circle, the function is not one-to-one. The classical way to describe this situation, is to say that the square root function (the nominal inverse of $f(z) = z^2$) is a **multiple valued** (or in this case bivalent) **function**. For example, the values of the complex square root $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = \sqrt{z}$ include $g(1) = 1$ and $g(1) = -1$. A more sophisticated approach to this situation is to actually make a new codomain called $\mathcal{R}_2$ consisting of both sheets, i.e., two copies of $\mathbb{C}$, indicated in Figure 2.2. For (almost) every point in the complex plane $\mathbb{C}$ there correspond two points in $\mathcal{R}_2$. For example, there is a point $w_0 = 1 = e^{0i}$ in the first, or principal, sheet of $\mathcal{R}_2$, and for this point $\sqrt{w_0} = 1$. But there is a second point $w_1 = 1 = e^{2\pi i}$ in the second sheet of $\mathcal{R}_2$, and for this point $\sqrt{w_1} = -1$.

The exceptional point is $w = 0$. There is only one point in $\mathcal{R}_2$ at $w = 0$, and this is called the **branch point**. Notice there is also only one, nicely defined, square root at the branch point: $\sqrt{0} = 0 \in \mathbb{C}$.

The domain/codomain $\mathcal{R}_2$ has a name, both as a general kind of thing and as a specific example of that kind of thing. The general kind of thing is called a **Riemann surface**, and $\mathcal{R}_2$ is the Riemann surface of $f(z) = z^2$. If you understand $\mathcal{R}_2$ correctly, then you should realize it is a kind of mind-blowing object. If you do not understand $\mathcal{R}_2$ much at all, then we should work a little harder—you can do it (fighting!).

Let us denote the two sheets of $\mathcal{R}_2$ by Σ_0 and Σ_1. Technically, we can write

$$\Sigma_0 = \{(z, 0) : z \in \mathbb{C}\backslash\{0\}\}.$$

Notice that Σ_0, if we ignore the second indexing coordinate, looks just like the punctured complex plane $\mathbb{C}\backslash\{0\}$. It may also be helpful to think about Σ_0 as the complex plane with (complex) numbers w having argument satisfying $0 \le \arg(w) < 2\pi$. The same is true of

$$\Sigma_1 = \{(z, 1) : z \in \mathbb{C}\backslash\{0\}\},$$

but for Σ_1 we can think of points $w \in \Sigma_1$ with $2\pi \le \arg(w) < 4\pi$. Then $\mathcal{R}_2 = \Sigma_1 \cup \Sigma_2 \cup \{0\}$. This description, so far, does not capture the most important feature of $\mathcal{R}_2$, which is how the two sheets are "glued" or "sewn" or joined together. They are joined together along a **branch cut**, which I sort of think should be called "two" branch cuts, but apparently no one agrees with me on this. Nevertheless, if we isolate the positive real axis in Σ_0 and the positive real axis in Σ_1, and we imagine cutting each plane along its positive real half line, then we have four "cut edges."

Pay close attention now: When "you" wander. . . or more properly if a point w wanders out of the first quadrant in Σ_0, if that point crosses the positive imaginary axis, it stays in Σ_0 just as you might expect. However, if the point w wanders out of the first quadrant in Σ_0 across the positive real axis, then you find yourself, or w

finds itself, in the fourth quadrant of Σ_1. This is illustrated in Figure 2.3, and this happens because the first quadrant of Σ_0 is "sewn together" with the fourth quadrant of Σ_1 along the positive real axis. Similarly, the fourth quadrant of Σ_0 is sewn to

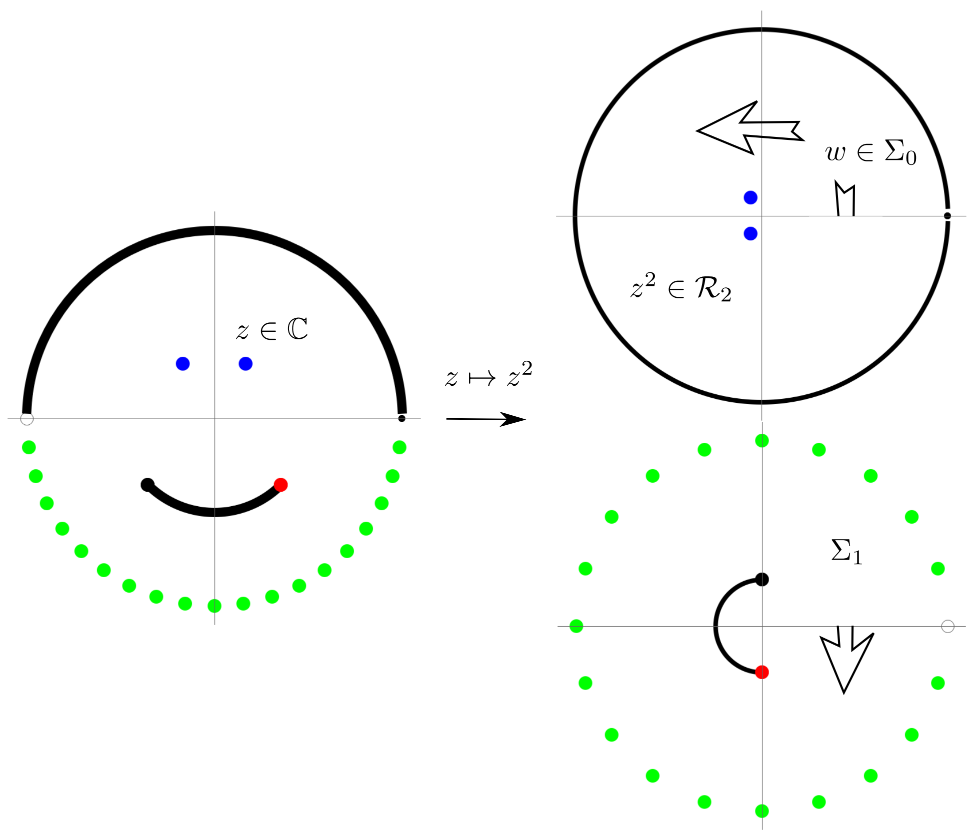

Figure 2.3: A point wandering around in the Riemann surface $\mathcal{R}_2$ for z^2. The branch cut is along the positive real axis. What happens if you exit the third quadrant of Σ_1 along the positive real axis? .

the first quadrant of Σ_1 along the positive real axis/axes. This is what I would call the "second branch cut," but traditionally, the entire construction is said to involve simply a branch cut along the positive real axis.

You should be able to picture now the principal square root of a complex number $w \in \mathbb{C}\backslash\{0\}$ as the image under the square root of $w \in \Sigma_0$. Of course, this is the principal square root with argument satisfying

$$0 \le \mathrm{Arg}(\sqrt{w}) < \pi.$$

The corresponding mapping $g : \mathbb{C} = \Sigma_0 \to \mathbb{C}$ by $g(w) = \sqrt{w}$ is called the principal square root (according to me). The map $g : \mathcal{R}_2 \to \mathbb{C}$ is the complex square root.

Exercise 2.6 Consider $f : \mathbb{C} \to \mathcal{R}_2$ by $f(z) = z^2$.

(a) Draw the radial lines

$$\mathcal{L}_n = \{t[\cos(n\pi/10) + i\sin(n\pi/10)] : t \ge 0\}$$

in red in $\mathbb{C}$ for $n = 1, 2, \ldots, 10$.

(b) Draw the images $f(\mathcal{L}_n)$ (in red) for $n = 1, 2, \ldots, 10$ in the Riemann surface $\mathcal{R}_2$.

(c) Draw the radial lines $\mathcal{L}_n$ in blue in $\mathbb{C}$ for $n = 11, 12, \ldots, 20$.

(d) Draw the images $f(\mathcal{L}_n)$ (in blue) for $n = 11, 12, \ldots, 20$ in the Riemann surface $\mathcal{R}_2$.

Exercise 2.7 Brown and Churchill designate the principal n-th root of $w = |w|e^{i\phi} \in \mathbb{C}\backslash\{0\}$ to be the complex number

$$\sqrt[n]{|w|}e^{i\phi/n}$$

when $\mathrm{Arg}(w) = \phi$ is taken with $-\pi < \phi \le \pi$. Construct a different Riemann surface for $f(z) = z^2$ so that the principal square root of Brown and Churchill is obtained as the inverse of f on a single sheet of the Riemann surface. Hint: Put the branch cut along the negative real axis.

We will come back to the complex squaring function, square root function, and the associated Riemann surface.

A more general class of "nice" complex valued functions of a complex variable are given by the **polynomials**. These look like the polynomials from, say, precalculus but they have complex coefficients in general, and we (may) want to think of them as mappings of the complex plane to itself (or into some Riemann surface):

$$P : \mathbb{C} \to \mathbb{C} \qquad \text{by} \qquad P(z) = \sum_{j=0}^{k} a_j z^j.$$

Just like polynomials with real (or rational or integer) coefficients, the **degree** of such a polynomial is k if the coefficient of the highest power term, a_k, is nonzero. Notice $f(z) = z^2$ is a polynomial mapping.

Exercise 2.8 What is the image of the open sector

$$\{re^{i\theta} : 0 < r \text{ and } 0 < \theta < \pi/6\}$$

under the polynomial map $f(z) = z^3$?

An even more general class of "nice" complex valued functions of a complex variable is given by the **rational** functions. Each such function has the form $q : \mathbb{C}\backslash\{z_1, z_2, \ldots, z_k\} \to \mathbb{C}$ where

$$q(z) = \frac{P(z)}{Q(z)} \qquad \text{with} \qquad P, Q \text{ polynomials}$$

and $\{z_1, z_2, \ldots, z_k\}$ is the collection of roots of Q, i.e., $\{z \in \mathbb{C} : Q(z) = 0\} = \{z_1, z_2, \ldots, z_k\}$. The function $f(z) = 1/z$ we introduced at the beginning of this discussion is one of these rational functions.

2.2 Complex limits and derivatives

These are my notes on (most of) sections 2.15-19 in BC.

I want to start with complex differentiation. Say $f : U \to \mathbb{C}$ has domain an open subset U of the complex plane. Say also that $z \in U$. Then the **derivative** of f at z is given by

$$f'(z) = \lim_{\zeta \to z} \frac{f(\zeta) - f(z)}{\zeta - z}$$

if this limit exists. A complex derivative of a function f, when it exists, can also be denoted by

$$\frac{df}{dz}.$$

Naturally, one might want to know what it means for a limit to exist. In this case, it means precisely the following: There is some fixed complex number $L = f'(z)$ such that given $\epsilon > 0$, there exists some $\delta > 0$ for which

$$0 < |\zeta - z| < \delta \qquad \Longrightarrow \qquad \left| \frac{f(\zeta) - f(z)}{\zeta - z} - L \right| < \epsilon.$$

More generally, if $z_0 \in U$ and $g : U\backslash\{z_0\} \to \mathbb{C}$, then we write

$$\lim_{z \to z_0} g(z) = w_0 \tag{2.3}$$

if for any $\epsilon > 0$, there is some $\delta > 0$ for which

$$0 < |z - z_0| < \delta \qquad \Longrightarrow \qquad |g(z) - w_0| < \epsilon.$$

There can only be one complex number w_0 for which the limiting assertion (2.3) holds. Note that if

$$\lim_{z \to z_0} g(z) = \tilde{w}_0,$$

then given any $\epsilon > 0$, we can find some $\delta > 0$ so that both

$$|g(z) - w_0| < \frac{\epsilon}{2} \qquad \text{and} \qquad |g(z) - \tilde{w}_0| < \frac{\epsilon}{2}$$

when $0 < |z - z_0| < \delta$. This means

$$|\tilde{w}_0 - w_0| < \epsilon. \tag{2.4}$$

Exercise 2.9 Obtain (2.4) carfully and in detail using the definition of what it means for a limit to exist and the triangle inequality.

Since limits are unique, if a function $f : U \to \mathbb{C}$ is differentiable at each point z in an open set $U \subset \mathbb{C}$, the value of the derivative defines a new function $f' : U \to \mathbb{C}$.

As usual, there are general results about limits:

$$\lim_{z \to z_0} [g(z) + h(z)] = \lim_{z \to z_0} g(z) + \lim_{z \to z_0} h(z).$$

$$\lim_{z \to z_0} [g(z)h(z)] = \left(\lim_{z \to z_0} g(z)\right)\left(\lim_{z \to z_0} h(z)\right).$$

$$\lim_{z \to z_0} \frac{g(z)}{h(z)} = \frac{\lim_{z \to z_0} g(z)}{\lim_{z \to z_0} h(z)}.$$

These should all be interpreted to mean that of the limits

$$\lim_{z \to z_0} g(z) \qquad \text{and} \qquad \lim_{z \to z_0} h(z) \tag{2.5}$$

exist, then the limit of the algebraic expression on the left will exist and be given by the algebraic expression of the limits on the right. And of course

$$\lim_{z \to z_0} h(z)$$

should not be zero for the last one.

Exercise 2.10 Find examples of two functions $g : \mathbb{C} \to \mathbb{C}$ and $h : \mathbb{C} \to \mathbb{C}$ and some $z_0 \in \mathbb{C}$ for which the limits in (2.5) both exist and $\lim_{z\to z_0} h(z) = 0$ and the following hold:

(a) $\lim_{z\to z_0} g(z)/h(z)$ exists

(b) $\lim_{z\to z_0} g(z)/h(z)$ does not exist.

Here is something a little (tiny bit) different: A function $f : U\backslash\{z\} \to \mathbb{C}$ with U open in $\mathbb{C}$ and $z \in U$ (as usual) satisfies

$$\lim_{\zeta\to z} f(\zeta) = w \in \mathbb{C}$$

if and only if

$$\lim_{\zeta\to z} \operatorname{Re} f(\zeta) = \operatorname{Re} w \qquad \text{and} \qquad \lim_{\zeta\to z} \operatorname{Im} f(\zeta) = \operatorname{Im} w. \tag{2.6}$$

The "if" part is just a consequence of the "sum of the limits" formula above. The "only if" part is the tiny bit different part. It is quite easy.

Exercise 2.11 Show that if $z = x + iy$ and

$$\lim_{\zeta\to z} f(\zeta) = w,$$

then

$$\lim_{(\xi,\eta)\to(x,y)} \operatorname{Re}(f)(\xi,\eta) = \operatorname{Re} w \qquad \text{and} \qquad \lim_{(\xi,\eta)\to(x,y)} \operatorname{Im}(f)(\xi,\eta) = \operatorname{Im} w.$$

We mentioned that $\mathbb{C}$ is a metric space, and the definitions/theory of limits of functions on a punctured open set in one metric space taking values in another metric are essentially the same as what we have described above. Similarly, the idea of a continuous function from one metric space to another can always be expressed in terms of limits. The simplest formulation is the following:

Definition 9 (continuity) Given and open set $U \subset \mathbb{C}$ with $z_0 \in U$ and a function $g : U \to \mathbb{C}$, we say g is **continuous at** z_0 if

$$\lim_{z\to z_0} g(z) = g(z_0).$$

The function $g : U \to bbc$ is **continuous**, or is in $C^0(U)$, if g is continuous at every $z_0 \in U$.

Exercise 2.12 (Exercises2.18.1-4 and 7) Show the following functions are continuous on all of $\mathbb{C}$:

(a) (imaginary part) $\text{Im} : \mathbb{C} \to \mathbb{C}$.

(b) (conjugation) $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = \overline{z}$.

(c) (complex square) $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$.

There is more about limits and continuity and stereographic projection and the point at ∞ (all good stuff) in BC, but I want to get (back to) differentiation. If, for example, $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$ is the complex square function, then

$$\frac{f(\zeta) - f(z)}{\zeta - z} = \zeta + z,$$

and so

$$\lim_{\zeta \to z} \frac{f(\zeta) - f(z)}{\zeta - z} = 2z$$

which looks totally "normal." This sort of "normal" thing happens for many complex functions including complex polynomials, products, quotients, and other functions. Expected "normal" differentiation happens for complex valued functions of a complex variable. But you should not imagine that complex differentiation is some kind of trivial generalization of regular differentiation from calculus I.

Let me see if I can illustrate this. Let $g : \mathbb{C} \to \mathbb{C}$ be complex conjugation given by $g(z) = \overline{z}$. If we were to take the corresponding function under the canonical bijection $\gamma : \mathbb{R}^2 \to \mathbb{C}$, we would get $G : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$G(x, y) = (x, -y).$$

This is a linear transformation of the plane, namely reflection about the x-axis. It has two component functions $u(x, y) = x$ and $v(x, y) = -y$. These have any partial derivatives you might wish to compute, and the overall transformation is differentiable in any way you might imagine. And yet...

$$\lim_{\mathbb{R} \ni h \to 0} \frac{g(z + h) - g(z)}{h} = 1$$

but

$$\lim_{\mathbb{R} \ni h \to 0} \frac{g(z + ih) - g(z)}{ih} = -1.$$

This means that for every $z \in \mathbb{C}$

$$\lim_{\zeta \to z} \frac{g(\zeta) - g(z)}{\zeta - z} \qquad \text{does not exist.}$$

The conjugation function $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = \overline{z}$ is not (complex) differentiable at any single point $z \in \mathbb{C}$.

Exercise 2.13 Show that if $f : U \to \mathbb{C}$ is complex differentiable, then $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ have first partial derivatives at every point in $\gamma^{-1}(U) \subset \mathbb{R}^2$.

For example, we can take $\zeta = z + h$ and $z = x + iy$ where $h \in \mathbb{R}$. Then

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x+h,y) - u(x,y)}{h} + i\frac{v(x+h,y) - v(x,y)}{h}.$$

By differentiability,

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(x+h,y) - u(x,y)}{h} = \operatorname{Re} f'(z)$$

and

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(x+h,y) - v(x,y)}{h} = \operatorname{Im} f'(z).$$

Of course, the existence of both partial derivatives for a real valued function of two real variables does not mean the function is continuous. You learned this in calculus III. . . or maybe you didn't.

Exercise 2.14 Give an example of a function $u : \mathbb{R}^2 \to \mathbb{R}$ for which both first partial derivatives exist at every point $(x, y) \in \mathbb{R}^2$, but u is not continuous at (at least) one point in $\mathbb{R}^2$.

But if f is differentiable at z, then f is continuous at z. Here is a proof:

Let $\epsilon > 0$. Then for $0 < |z - z_0|$ we can write

$$\begin{aligned}
|f(z) - f(z_0)| &= \left| \frac{f(z) - f(z_0)}{z - z_0} \right| \; |z - z_0| \\
&= \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) + f'(z_0) \right| \; |z - z_0| \\
&\leq \left(\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| + |f'(z_0)| \right) \; |z - z_0|.
\end{aligned}$$

By differentiability, we can take/find $\delta > 0$ so that

$$0 < |z - z_0| < \delta \qquad \Longrightarrow \qquad \begin{cases} \left| \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \dfrac{\epsilon}{2}, \\ |z - z_0| < 1, \text{ and} \\ |z - z_0| < \dfrac{\epsilon}{2(|f'(z_0)| + 1)}. \end{cases}$$

Thus, if $0 < |z - z_0| < \delta$, then

$$|f(z) - f(z_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and this is what it means for f to be continuous at z_0. □

We have now covered some basic facts about differentiability and continuity. I guess I'm content to consider stereographic projection and limits at infinity.

Stereographic projection and limits at infinity

Consider the two sets

$$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\} \qquad \text{and} \qquad \mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

These sets are called the **extended complex plane** and the **two-sphere** respectively. Much like there is a vector space isomorphism $\gamma : \mathbb{R}^2 \to \mathbb{C}$ translating algebraic vector space properties, there is a one-to-one correspondence $\sigma : \mathbb{S}^2 \to \mathbb{C}_\infty$ called **stereographic projection** translating analytic (and topological) properties. For the moment, we will focus primarly on two aspects of this correspondence:

1. We can map certain open sets in $\mathbb{R}^2$ to open sets in $\mathbb{C}_\infty$ to get "open neighborhoods of ∞." (This is topology.)

2. Using these open sets, we can define for $g : U \to \mathbb{C}_\infty$ where U is open in $\mathbb{C}$ limits of the form
$$\lim_{z \to \zeta} g(z) = \omega$$
where $\zeta, \omega \in \mathbb{C}_\infty$. (This may be considered also topological, but we will use it in definitively analytic contexts.)

The idea is very easy: A **nieghborhood of ∞ in** $\mathbb{C}$ is an open set U in $\mathbb{C}$ such that for some $R > 0$,

$$\mathbb{C}\backslash B_R(0) \subset U.$$

Such a set U is then also a **punctued neighborhood of** ∞ in $\mathbb{C}_\infty$, and a set of the form $U \cup \{\infty\}$ is a full **neighborhood of** ∞ **in** $\mathbb{C}_\infty$.

Exercise 2.15 Draw pictures of these three kinds of neighborhoods.

As mathematicians we have to make this more complicated by explaining it with formulas and functions to make it more precise and "easier."

They key player, **stereographic projection**, as mentioned above, is the function $\sigma : \mathbb{S}^2 \to \mathbb{C} \cup \{\infty\}$ given by

$$\sigma(x, y, z) = \begin{cases} \infty, & \text{if } z = 1 \\ \dfrac{x}{1-z} + i\dfrac{y}{1-z}, & \text{if } z < 1. \end{cases}$$

Both $\mathbb{S}^2$ and $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ are called the **Riemann sphere** in this context. Here are some things you can do with them:

1. There is a natural topology (of open sets) on $\mathbb{S}^2$ obtained by intersecting open sets in $\mathbb{R}^3$ with $\mathbb{S}^2$.

2. In $\mathbb{S}^2$ the north pole $N = (0, 0, 1)$ looks pretty much like any other point. (You can notice this.)

3. Given a real or complex valued function $G : \mathbb{S}^2 \to F$ where F denotes one of the fields $\mathbb{R}$ or $\mathbb{C}$, it is quite easy to talk about/understand the continuity of G at a point $(x_0, y_0, z_0) \in \mathbb{S}^2$: For any $\epsilon > 0$, there is some $\delta > 0$ such that

   $$\left.\begin{array}{l} \|(x, y, z) - (x_0, y_0, z_0)\| < \delta \\ (x, y, z) \in \mathbb{S}^2 \end{array}\right\} \implies |G(x, y, z) - G(x_0, y_0, z_0)| < \epsilon,$$

   That is,

   $$\lim_{\mathbb{S}^2 \ni (x,y,z) \to (x_0, z_0, z_0)} G(x, y, z) = G(x_0, y_0, z_0).$$

   Here we can use (and have used) the **Euclidean norm** on $\mathbb{R}^3$

   $$\|(x, y, z)\| = \sqrt{x^2 + y^2 + z^2}$$

   restricted to $\mathbb{S}^3$.

4. Stereographic projection also gives a correspondence between functions on $\mathbb{C} \cup \{\infty\}$ and $\mathbb{S}^2$:

$$G(\mathbf{x}) = g \circ \sigma(\mathbf{x})$$
$$g(z) = G \circ \sigma^{-1}(z),$$

or between functions $g : \mathbb{C} \to \mathbb{C}$ and functions $G = g \circ \sigma : \mathbb{S}^2 \backslash \{N\} \to \mathbb{C}$, **or** between functions $g : U \to \mathbb{C}$ and functions $G = g \circ \sigma : \sigma^{-1}(U) \backslash \{N\} \to \mathbb{C}$.

In the last case, $\sigma^{-1}(U) \backslash \{N\}$ may be a "nice" punctured neighborhood of $N \in \mathbb{S}^2$. So

$$\lim_{\mathbb{S}^2 \ni \mathbf{x} \to N} G(\mathbf{x})$$

may make perfectly good (topological/analytic) sense.

5. Making

$$\{\sigma^{-1}(V) : V \text{ is open in } \mathbb{S}^2\}$$

a topology on $\mathbb{C} \cup \{\infty\}$ makes good sense.

6. Defining

$$\lim_{z \to \infty} g(z)$$

in terms of

$$\lim_{\mathbf{x} \to N} G(\mathbf{x})$$

also makes good sense.

Aside from all these observations, there is sort of one more important thing to note: When you want to do analysis on a function $f : U \to \mathbb{C}$ defined on a puctured neighborhood U of ∞ in $\mathbb{C}$ or a neighborhood of ∞ in $\mathbb{C} \cup \{\infty\}$ and ask questions like

1. Is f continuous at ∞?

2. Is f **differentiable at** ∞?

You should consider the function $g : V \to \mathbb{C}$ given by

$$g(\zeta) = f\left(\frac{1}{\zeta}\right)$$

where

$$V = \left\{ \frac{1}{z} : z \in U \backslash \{\infty\} \right\}.$$

Note that V is a punctured neighborhood of $0 \in \mathbb{C}$. If

$$f(\infty) = \lim_{z \to \infty} f(z),$$

then one can also consider

$$g(\zeta) = \begin{cases} f\left(\frac{1}{\zeta}\right), & \zeta \in V \backslash \{0\} \\ f(\infty), & \zeta = 0. \end{cases}$$

We say, for example, that f is differentiable at $z = \infty$ if g is differentiable at $\zeta = 0$.

2.3 Differentiability (continued)

Let's return to differentiability at finite points, i.e., given an open set $U \subset \mathbb{C}$ and a function $f : U \to \mathbb{C}$ we are interested in the situation where f is (complex) differentiable at a point $z_0 \in U$ and especially when f is complex differentiable at all points $z_0 \in U$. Basically, one can say the familar differentiation rules from calculus hold:

Linearity

If $f, g : U \to \mathbb{C}$ are differentiable and $\alpha, \beta \in \mathbb{C}$, then $af + bg$ is differentiable with

$$\frac{d}{dz}[\alpha f + \beta g] = \alpha f' + \beta g'.$$

Power Rule(s)

$$\frac{d}{dz} z^n = n z^{n-1} \qquad \text{on all of } \mathbb{C} \text{ for } n = 1, 2, 3, \ldots.$$

If f is constant, e.g., $f(z) = z^0$, then $f' \equiv 0$. The functions $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ by $f(z) = 1/z^2$ are differentible on the punctured plane $\mathbb{C}\backslash\{0\}$ with

$$\frac{d}{dz}\frac{1}{z^n} = -\frac{n}{z^{n+1}} \qquad \text{for } n = 1, 2, 3, \ldots.$$

Putting the power rules (for positive exponents) together with linearity gives the derivatives of polynomials, which behave as one would expect (at least concerning the formulas) from calculus.

Exercise 2.16 (Legendre polynomials) The real Legendre polynomials can be defined as follows $P_0(x) \equiv 1$, and for each $n = 1, 2, 3, \ldots$, P_n is the degree n polynomial for which the following hold

(i) $P_n(1) = 1$.

(ii)

$$\int_{-1}^{1} P_n(x) P_j(x)\, dx = 0 \qquad \text{for } 0 \leq j < n.$$

Plot the first few (real) Legendre polynomials on the interval $[-1, 1]$, and show they are given by

$$P_n(x) = \frac{1}{n! 2^n} \frac{d^n}{dx^n} (x^2 - 1)^n \qquad \text{for } n = 1, 2, 3, \ldots.$$

The complex Legendre polynomials $P_n = P_n(z)$ have (and satisfy) the same formulas as polynomial functions on the complex plane.

Product rule

If $f, g : U \to \mathbb{C}$ are differentiable, then $fg : U \to \mathbb{C}$ is differentiable with

$$(fg)' = f'g + fg'.$$

Quotient rule

If $f, g : U \to \mathbb{C}$ are differentiable and $g \neq 0$, then f/g is differentiable and

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

Chain rule

If U and V are open subsets of $\mathbb{C}$ with $f : U \to \mathbb{C}$ and $g : V \to \mathbb{C}$ differentiable functions for which $g(V) \subset U$, then the **compostion** $g \circ f : U \to \mathbb{C}$ given by

$$g \circ f(z) = g(f(z))$$

is differentiable and

$$\frac{d}{dz}(g \circ f) = (g' \circ f)\frac{df}{dz}.$$

Again, I want to emphasize that the fact that all these "calculus" rules for differentiation apply to complex differentiable functions does not at all mean the complex differentiable functions are directly comparable to differentiable functions $f : \mathbb{R} \to bbr$ on the real field. Though the rules still work, there are, in a certain sense, **far fewer** complex differentiable functions either than differentiable functions $f : \mathbb{R} \to \mathbb{R}$ or differentiable functions $f : \mathbb{R}^2 \to \mathbb{R}^2$. There are just fewer complex differentiable functions to start with. Some idea of which functions have "gone missing" may be appreciated by considering $f(x) = x^2 = |x|^2$ and $f(z) = |z|^2$.

We will now start to explore how special complex differentiable functions really are using the Cauchy-Riemann equations.

2.4 The Cauchy-Riemann Equations

If $f : U \to \mathbb{C}$ is complex differentiable on an open set $U \subset \mathbb{C}$, then

$$f'(z) = \lim_{\zeta \to z} \frac{f(\zeta) - f(z)}{\zeta - z}. \tag{2.7}$$

Taking the limit using values $\zeta = z + h$ with $h \in \mathbb{R}$, we find

$$\begin{aligned} f'(z) &= \lim_{h \to 0} \frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x,y)}{h} \\ &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y). \end{aligned} \tag{2.8}$$

Thus, we have an interesting formula for the derivative of $f = u + iv$ in terms of the real partial derivatives of the real and imaginary parts u and v:

$$f' = u_x + iv_x.$$

Similarly, taking the limit giving the derivative $f'(x)$ using valued $\zeta = z + ih$, we obtain a second formula:

$$f' = v_y - iu_y.$$

Exercise 2.17 (Cauchy-Riemann equations) Carry out the details of taking the limit (2.7) using values $\zeta = z + ih$ with $\mathbb{R} \ni h \to 0$.

Equating the two expressions for f' we obtain the **Cauchy-Riemann equations**:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}. \end{cases} \tag{2.9}$$

This is a system of two (real) partial differential equations.

There is a kind of converse:

Theorem 3 Let $u, v : \mathcal{U} \to \mathbb{R}$ where $\mathcal{U}$ is an open subset of $\mathbb{R}^2$ and assume u and v have **continuous first order partial derivatives**[4]

$$\frac{\partial u}{\partial x}, \ \frac{\partial u}{\partial y}, \ \frac{\partial v}{\partial x}, \ \frac{\partial v}{\partial y} \ : \ \mathcal{U} \to \mathbb{R}$$

satifying the Cauchy-Riemann equations on U. Then the complex function $f : U \to \mathbb{C}$ defined by $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$ and $U = \gamma(\mathcal{U})$ are given by the canonical identifications is complex differentiable.

Before I prove Theorem 3, let me mention that the following may be a good question to ask:

> Back in (2.8) how do we know the partial derivatives of the functions $u = \text{Re}(f)$ and $v = \text{Im}(f)$ exist?

The assertion that

$$\lim_{h\to 0} \frac{u(x+h, y) - u(x, y)}{h} \qquad \text{and} \qquad \lim_{h\to 0} \frac{v(x+h, y) - v(x, y)}{h}$$

exist may be justified as follows: Let $\epsilon > 0$. Then there is some $\delta > 0$ for which

$$\left| \frac{f(z+h) - f(z)}{h} - f'(z) \right| < \epsilon \qquad \text{for} \qquad 0 < |h| < \delta.$$

[4]As a matter of (standard) notation, the collection of all functions with domain an open set $\mathcal{U} \subset \mathbb{R}^2$ having continuous first partial derivatives is denoted by $C^1(\mathcal{U})$. Thus, the statement of Theorem 3 may be shortened by writing simply $u, v \in C^1(\mathcal{U})$. You may also be interested to know how to pronounce/say "$C^1(\mathcal{U})$." When you say it, it should sound like this: "sea one of you" or perhaps "see one of you." The collection of continuous functions of two variables on the same set $\mathcal{U}$ is denoted by $C^0(\mathcal{U})$. Can you guess how to express this notation verbally? Answer: "see zero of you." It is true that $C^1(\mathcal{U}) \subset C^0(\mathcal{U})$. Can you prove it?

This is what it means for f to be complex differentiable. As written above, this also gives

$$\left| \frac{u(x+h,y)-u(x,y)}{h} - \mathrm{Re}[f'(z)] + i\left(\frac{v(x+h,y)-v(x,y)}{h} - \mathrm{Im}[f'(z)]\right)\right| < \epsilon$$

when $0 < |h| < \delta$. Therefore, when $0 < |h| < \delta$ we know for example

$$\begin{aligned}\left|\frac{u(x+h,y)-u(x,y)}{h} - \mathrm{Re}[f'(z)]\right| \le \Bigg| & \frac{u(x+h,y)-u(x,y)}{h} - \mathrm{Re}[f'(z)] \\ & +i\left(\frac{v(x+h,y)-v(x,y)}{h} - \mathrm{Im}[f'(z)]\right)\Bigg| \\ < \epsilon. &\end{aligned}$$

Note: We are not using the triangle inequality here but rather the inequality $|\mathrm{Re}(z)| \le |z|$ for every complex number z, which is a different thing.[5]

In any case, this means

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(x+h,y)-u(x,y)}{h} \qquad \text{exists,}$$

and in fact as we see this partial derivative has the value $\mathrm{Re}[f'(z)]$. The other partial derivative of $u = \mathrm{Re}(f)$ and both first partials of $v = \mathrm{Im}(f)$ may be seen to exist similarly.

Proof of Theorem 3: It is enough to show

$$\lim_{\zeta\to z} \frac{f(\zeta)-f(z)}{\zeta - z} = u_x(x,y) + iv_x(x,y)$$

where as usual $z = x + iy$. Letting $\zeta - z = h + ik$ we can write

$$\begin{aligned}\left|\frac{f(\zeta)-f(z)}{\zeta-z} - [u_x(x,y)+iv_x(x,y)]\right| & \\ = \Bigg| & \frac{u(x+h,y+k)-u(x,y)}{h+ik} - u_x(x,y) \\ & +i\left(\frac{v(x+h,y+k)-v(x,y)}{h+ik} - v_x(x,y)\right)\Bigg|. \qquad (2.10)\end{aligned}$$

[5] In particular, this inequality follows from the definition of the absolute value/modulus of a complex number; see (1) and (3) on page 9, Chapter 1, Section 4 of BC.

Admittedly, the expressions on the right are not exactly what we'd like to see, but we are, in some sense, stuck with them. Do you know what we/I would like to see? I'd like to see differences like

$$\frac{u(x+h,y)-u(x,y)}{h} - u_x(x,y) \qquad \text{and} \qquad \frac{v(x+h,y)-v(x,y)}{h} - v_x(x,y).$$

I know I can make these small by making h small. In view of the Cauchy-Riemann equations, expressions approximating $u_y(x,y)$ and $v_y(x,y)$, i.e., differences like

$$\frac{u(x,y+k)-u(x,y)}{k} - u_y(x,y) \qquad \text{and} \qquad \frac{v(x,y+k)-v(x,y)}{k} - v_y(x,y)$$

would also be nice to see. The good news is that expressions looking vaguely like these are present, so maybe there is some hope. Let's see what we can come up with (complex) algebraically.

Okay. I've worked out what to do on scratch paper, and if I'm going to present it here, it strikes me that I want to draw your attention to three components of my approach:

1. I'm going to try to be somewhat (notationally) organized. This is partially because I'm typing this up, and the format does not lend itself to long expressions that fill up a tabloid (11" x 17") piece of paper.

2. I'm going to make an observation or two. In a certain sense, these may be viewed as technicalities in the background, but they are important.

3. I'm going to need a couple standard "tricks" or techniques from analysis, including a certain application of the familiar mean value theorem from calculus.

With these components in mind, let's give it go. Let

$$w = \frac{f(\zeta)-f(z)}{\zeta - z}$$

denote the complex difference quotent above which we have expressed as

$$w = \frac{u(x+h,y+k)-u(x,y)}{h+ik} + i\,\frac{v(x+h,y+k)-v(x,y)}{h+ik}.$$

Perhaps we can take, as a first step, finding the real and imaginary parts of the difference quotient w. As a second expression of notational organization, lets set

$$A_0 = u(x+h,y+k)-u(x,y) \qquad \text{and} \qquad B_0 = v(x+h,y+k)-v(x,y)$$

and note that these are real numbers. Thus,

$$w = \frac{1}{h^2 + k^2}\,[hA_0 + kB_0 + (hB_0 - kA_0)i] \tag{2.11}$$

where we are assuming $h^2+k^2 > 0$, but we are allowed, at least eventually, to assume $0 < h^2 + k^2 < \delta$ for some $\delta > 0$ we get to choose. We'll come back to this point after the algebraic manipulations are over.

Here is a first trick/technique: When I see an expression like A_0, I'm going to write

$$A_0 = u(x + h, y + k) - u(x, y + k) + u(x, y + k) - u(x, y) = A_1 + A_2 \tag{2.12}$$

where

$$A_1 = u(x + h, y + k) - u(x, y + k) \tag{2.13}$$

and

$$A_2 = u(x, y + k) - u(x, y). \tag{2.14}$$

The advantage of these differences is that only one component is changing (in each one). Let's focus first, for example, on the quantity A_1. I couple this (focus) with an important observation: Because $\mathcal{U}$ is an open subset of $\mathbb{R}^2$, I know I can take $h^2 + k^2$ small enough, say

$$|\zeta - z| = \sqrt{h^2 + k^2} < \delta_0,$$

so that all the points

$$x + t + i(y + k) \qquad \text{for} \qquad |t| \le |h| \tag{2.15}$$

lie in U or correspondingly

$$(x + t, y + k) \in \mathcal{U} \qquad \text{for} \qquad |t| \le |h|. \tag{2.16}$$

Exercise 2.18 Draw pictures illustrating (2.15) and (2.16).

Now, let me assume for a moment that $|h| > 0$. Then $g : [-|h|, |h|] \to \mathbb{R}$ by

$$g(t) = u(x + t, y + k)$$

for k (and $h \neq 0$) fixed is a real valued function (well) defined and continuous on the interval $[-|h|, |h|]$ and continuously differentiable (at least) on the open interval $(-|h|, |h|)$. In the usual notation of analysis:

$$g \in C^0[-|h|, |h|] \cap C^1(-|h|, |h|).$$

These, you may remember, are adequate conditions on g to apply the mean value theorem to conclude

$$\frac{g(h)-g(0)}{h} = g'(t_*) \qquad \text{for some } t_* \in (-|h|, |h|).$$

Multiplying through by h and expressing the value(s) of g and its derivative in terms of the function u, we have a point $x_* = x + t_*$ between x and $x + h$ for which

$$A_1 = g(h) - g(0) = hg'(t_*) = h\frac{\partial u}{\partial x}(x_*, y+k). \tag{2.17}$$

Notice finally that even when $h = 0$ the equality (2.17) still holds. Therefore, we can substitute in general to find

$$A_0 = A_1 + A_2 = h\frac{\partial u}{\partial x}(x_*, y+k) + A_2$$

and

$$hA_0 = h^2\frac{\partial u}{\partial x}(x_*, y+k) + hA_2 \qquad \text{and} \qquad kA_0 = hk\frac{\partial u}{\partial x}(x_*, y+k) + kA_2.$$

These expressions are not in a fully useful form quite yet, but notice the appearance of these quantities in the expression (2.11) for the difference quotient w indicates pretty clearly where we are going.

Exercise 2.19 Use the mean value theorem applied to an approriate function of one real variable to conclude

$$A_2 = k\frac{\partial u}{\partial y}(x, y_*)$$

for some y_* between y and $y + k$ so that

$$hA_0 = h^2\frac{\partial u}{\partial x}(x_*, y+k) + hk\frac{\partial u}{\partial y}(x, y_*)$$

and

$$kA_0 = hk\frac{\partial u}{\partial x}(x_*, y+k) + k^2\frac{\partial u}{\partial y}(x, y_*).$$

Using the same approach, we can find other values ξ_* between x and $x + h$ and η_* between y and $y + k$ so that

$$B_0 = h\frac{\partial v}{\partial x}(\xi_*, y+k) + k\frac{\partial v}{\partial y}(x, \eta_*).$$

It follows that

$$\begin{aligned}\mathrm{Re}(w) &= \frac{1}{h^2+k^2}[hA_0 + kB_0] \\ &= \frac{1}{h^2+k^2}\left[h^2\frac{\partial u}{\partial x}(x_*, y+k) + hk\left(\frac{\partial u}{\partial y}(x, y_*) + \frac{\partial v}{\partial x}(\xi_*, y+k)\right) + k^2\frac{\partial v}{\partial y}(x, \eta_*)\right]\end{aligned}$$

and

$$\begin{aligned}\mathrm{Im}(w) &= \frac{1}{h^2+k^2}[hB_0 - kA_0] \\ &= \frac{1}{h^2+k^2}\left[h^2\frac{\partial v}{\partial x}(\xi_*, y+k) + hk\left(\frac{\partial v}{\partial y}(x, \eta_*) - \frac{\partial u}{\partial x}(x_*, y+k)\right) - k^2\frac{\partial u}{\partial y}(x, y_*)\right].\end{aligned}$$

Let us apply to these expressions the Cauchy-Riemann equations, replacing the partial derivatives with respect to y with partial derivatives with respect to x only:

$$\mathrm{Re}(w) = \frac{1}{h^2+k^2}\left[h^2\frac{\partial u}{\partial x}(x_*, y+k) + hk\left(\frac{\partial v}{\partial x}(\xi_*, y+k) - \frac{\partial v}{\partial x}(x, y_*)\right) + k^2\frac{\partial u}{\partial x}(x, \eta_*)\right]$$

and

$$\mathrm{Im}(w) = \frac{1}{h^2+k^2}\left[h^2\frac{\partial v}{\partial x}(\xi_*, y+k) + hk\left(\frac{\partial u}{\partial x}(x, \eta_*) - \frac{\partial u}{\partial x}(x_*, y+k)\right) + k^2\frac{\partial v}{\partial x}(x, y_*)\right].$$

Returning to the beginning of our proof and the expression (2.10) for $|w - (u_x + iv_x)|$

in particular, we have shown

$$\begin{aligned}
|\mathrm{Re}(w) - u_x| &= \left| \frac{1}{h^2+k^2} \left[h^2 \frac{\partial u}{\partial x}(x_*, y+k) \right.\right. \\
&\qquad + hk \left(\frac{\partial v}{\partial x}(\xi_*, y+k) - \frac{\partial v}{\partial x}(x, y_*) \right) \\
&\qquad \left.\left. + k^2 \frac{\partial u}{\partial x}(x, \eta_*) \right] - \frac{\partial u}{\partial x}(x,y) \right| \\
&= \frac{1}{h^2+k^2} \left| h^2 \left(\frac{\partial u}{\partial x}(x_*, y+k) - \frac{\partial u}{\partial x}(x,y) \right) \right. \\
&\qquad + hk \left(\frac{\partial v}{\partial x}(\xi_*, y+k) - \frac{\partial v}{\partial x}(x, y_*) \right) \\
&\qquad \left. + k^2 \left(\frac{\partial u}{\partial x}(x, \eta_*) - \frac{\partial u}{\partial x}(x,y) \right) \right| \\
&\le \frac{h^2}{h^2+k^2} \left| \frac{\partial u}{\partial x}(x_*, y+k) - \frac{\partial u}{\partial x}(x,y) \right| \\
&\qquad + \frac{|hk|}{h^2+k^2} \left| \frac{\partial v}{\partial x}(\xi_*, y+k) - \frac{\partial v}{\partial x}(x, y_*) \right| \\
&\qquad + \frac{k^2}{h^2+k^2} \left| \frac{\partial u}{\partial x}(x, \eta_*) - \frac{\partial u}{\partial x}(x,y) \right| \\
&\le \left| \frac{\partial u}{\partial x}(x_*, y+k) - \frac{\partial u}{\partial x}(x,y) \right| \\
&\qquad + \frac{1}{2} \left| \frac{\partial v}{\partial x}(\xi_*, y+k) - \frac{\partial v}{\partial x}(x, y_*) \right| \\
&\qquad + \left| \frac{\partial u}{\partial x}(x, \eta_*) - \frac{\partial u}{\partial x}(x,y) \right| \qquad (2.18)
\end{aligned}$$

by virtue of the triangle inequality and the fact that each of the fractions

$$\frac{h^2}{h^2+k^2}, \quad \frac{2|hk|}{h^2+k^2}, \quad \text{and} \quad \frac{k^2}{h^2+k^2}$$

is less than or equal to 1. Each of the three differences of partial derivatives in (2.18) can be made small by making $|\zeta - z| = \sqrt{h^2+k^2}$ small. Specifically, given $\epsilon > 0$, there is some $\delta_1 > 0$ for which

$$0 < |\zeta - z| < \min\{\delta_0, \delta_1\} \qquad \text{impies} \qquad |\,\mathrm{Re}(w) - u_x(x,y)| < \frac{\epsilon}{\sqrt{2}}.$$

Exercise 2.20 Show there exists some $\delta_2 > 0$ for which

$$0 < |\zeta - z| < \min\{\delta_0, \delta_2\} \qquad \text{impies} \qquad |\operatorname{Im}(w) - v_x(x,y)| < \frac{\epsilon}{\sqrt{2}}.$$

Putting these estimates together we conclude that for $0 < |\zeta - z| < \delta = \min\{\delta_0, \delta_1, \delta_2\}$ there holds

$$\begin{aligned}\left|\frac{f(\zeta) - f(z)}{\zeta - z} - [u_x(x,y) + iv_x(x,y)]\right| & \\ &= \sqrt{|\operatorname{Re}(w) - u_x(x,y)|^2 + |\operatorname{Im}(w) - v_x(x,y)|^2} \\ &< \epsilon. \qquad \square\end{aligned}$$

Exercise 2.21 Redo the entire proof of Theorem 3 incrementing along the alternative horizontal and vertical lines in U, writing for example

$$u(x+h, y+k) - u(x,y) = u(x+h, y+k) - u(x+h, y) + u(x+h, y) - u(x,y)$$

and/or

$$v(x+h, y+k) - v(x,y) = v(x+h, y+k) - v(x+h, y) + v(x+h, y) - v(x,y)$$

instead of

$$u(x+h, y+k) - u(x,y) = u(x+h, y+k) - u(x, y+k) + u(x, y+k) - u(x,y)$$

and

$$v(x+h, y+k) - v(x,y) = v(x+h, y+h) - v(x, y+k) + u(x, y+k) - u(x,y).$$

2.5 Complex differentiability and the domain

There are some important and interesting topics, and some useful terminology and examples, in sections 25-29 of Chapter 2 of BC. My inclination, however, is to hit some of the (necessary) high points and leave most of it for later. One reason for this, is that some results from later are quoted, and I don't think one can get, at this point, a clear view of why those results hold. The basic topics fall under the heading of **analytic continuation and the reflection principle**. The underlying idea is that if two complex differentiable functions $f, g : U \to \mathbb{C}$ have the same values on a collection of distinct points $\{z_j\}_{j=1}^{\infty} \subset U$, where U is an open subset of $\mathbb{C}$ as usual which is **connected**, **and** the the points $\{z_j\}_{j=1}^{\infty}$ have an accumulation point $z_0 \in U$, then $f \equiv g$ on U.

The concept of a set $U \subset \mathbb{C}$ being connected is relatively straightforward to explain at this point, though I don't think we/I have done it. The basic reason two such complex differentiable functions end up being the same function, however, follows from a relatively deep (and very important) fact about complex differentiable functions which really is only taken up in Chapter 4 of BC. So I think we/you should work hard on Chapter 4 and then come back to these topics. And of course, to do that we/you need to work hard on Chapter 3 to get to Chapter 4. In any case, for now, I will try to give some kind of brief overview/summary of the rest of Chapter 2 with emphasis on a very few points.

§ 25

Three things to note here:

1. (terminology) If $f : U \to \mathbb{C}$ is complex differentiable and $U = \mathbb{C}$, then f is said to be **entire**.

2. (terminology) If U is an open subset of $\mathbb{C}$ and $f : U \backslash \{z_0\} \to \mathbb{C}$, then z_0 is said to be an **isolated singularity**.

3. (connected sets) The topic of connectedness is not really addressed properly anywhere in Chapter 1 or Chapter 2 of BC (and probably not anywhere in the book). It is a relatively easy topic however, and it is pretty important (I think), so let me discuss it properly now.

Let's first start with open sets. This is, in fact, I think the only case mentioned (and used) in BC.

Definition 10 An open set $U \subset \mathbb{C}$ is **connected** if U is not the union of two nonempty, disjoint, open subsets.

Of course, this definition is worth thinking about a little bit. It says, for example, that if $U_1, U_2 \subset U$ satisfying

(i) $U = U_1 \cup U_2$,

(ii) U_1, U_2 are open, and

(iii) $U_1 \cap U_2 = \phi$,

then one of the sets U_1 or U_2 is empty.

Exercise 2.22 Rephrase the definition of connected as I have done giving various assumptions about the sets U_1 and U_2 and then concluding what must be true about U_1 and U_2. I guess there are three main obvious rephrasings assuming U_1 and U_2 are nonempty.

Here is a different formulation:

Definition 11 An open set $U \subset \mathbb{C}$ is **path connected** if given any two points $z, w \in U$, there is a path in U connecting z to w.

A **path** is a continuous function $\gamma : [a, b] \to \mathbb{C}$ where $a, b \in \mathbb{R}$ with $a < b$ so that $[a, b]$ is some closed interval. We say a **path** γ **is in** U if $\gamma(t) \in U$ for all $t \in [a, b]$, or equivalently $\gamma : [a, b] \to U$. Such a path **connects** z **to** w if $\gamma(a) = z$ and $\gamma(b) = w$.

One special kind of path (in $\mathbb{C}$) is a **straight line path** from one point $z \in \mathbb{C}$ to another points $w \in \mathbb{C}$:

$$\gamma : [0, 1] \to \mathbb{C} \qquad \text{by} \qquad \gamma(t) = (1 - t)z + tw.$$

If one **concatenates** straightline paths, then one obtains what is called a **polygonal path**. For example, if $z, \zeta, w \in \mathbb{C}$, then we can take the straight line paths

$$\gamma_1(t) = (1 - t)z + t\zeta \qquad \text{and} \qquad \gamma_2(t) = (1 - t)\zeta + tw$$

and concatenate them by defining $\gamma : [0, 2] \to \mathbb{C}$ with

$$\gamma(t) = \begin{cases} \gamma_1(t), & t \in [0, 1] \\ \gamma_2(t - 1), & t \in [1, 2]. \end{cases}$$

You might notice that the domain interval $[a, b]$ for a path is not really very significant in defining the path. Also, the particular continuous dependence of a path is not very significant. For example, the paths

$$\alpha : [0, 1] \to \mathbb{C} \qquad \text{by} \qquad \alpha(t) = (1 - t)z + tw$$

and

$$\beta : [0, 1] \to \mathbb{C} \qquad \text{by} \qquad \beta(t) = (1 - t^2)z + t^2 w$$

are, in some sense, not so different. One can move in the direction of properly distinguishing between (and especially not distinguishing between) various paths by defining various equivalence relations or demanding that all paths be considered on a single standard closed interval, say $[0, 1]$, but we really don't have too much need to get into that at the moment. We may, at some point, want to assert the uniqueness of some path from one point to another, in some way shape or form, but hopefully we can cross that path when we come to it.

Let's see if I can state the main results with what we have so far.

Theorem 4 An open set $U \subset \mathbb{C}$ is connected if and only if U is path connected.

Theorem 5 An open set $U \subset \mathbb{C}$ is path connected if and only if each pair of points $z, w \in U$ can be connected to one another by a polygonal path in U.

Exercise 2.23 Prove Theorem 4.

Exercise 2.24 Prove Theorem 5.

In closing, the concept of a set being connected may be extended to subsets of $\mathbb{C}$ which are not open: A set $A \subset \mathbb{C}$ is connected if whenever U_1 and U_2 are open subsets of $\mathbb{C}$ with $A = (U_1 \cap A) \cup (U_2 \cap A)$, then one of the following must hold

(i) $(U_1 \cap A) \cap (U_2 \cap A) \neq \phi$ or

(ii) one of the sets $U_1 \cap A$ or $U_2 \cap A$ is empty.

For sets that are not open, being connected is not equivalent to being path connected in general.

Exercise 2.25 Define what it means for any set $A \subset \mathbb{C}$ to be path connected, and prove that any path connected set is connected.

Exercise 2.26 Give and example of a connected set $A \subset \mathbb{C}$ which is connected but not path connected.

I don't think we really need these more general defintions concerning connectedness, but they are fun.

Even more generally, the concept of a set being connected can be extended to any **topological space**. Rather than get into the details of what it means to be a topological space (which is also pretty easy, but we need to get to Chapters 3, 4, and

5 of BC) let me just say that a topological space is a set in which one knows what it means for a subset to be open. And let me give an example: For any set $A \subset \mathbb{C}$ we can declare a subset $V \subset A$ to be open if $V = A \cap U$ for some open set $U \subset \mathbb{C}$. The collection of sets

$$\{U \cap A : U \text{ is open in } \mathbb{C}\}$$

is called the **relative topology** on A; this is the collection of subsets of A considered/known to be open.

Exercise 2.27 Knowing now what it means to be open in A, go back and rephrase or "simplify" the definition of what it means for A to be connected.

Having given this discussion of connected sets, we can prove something:

Theorem 6 If $U \subset \mathbb{C}$ is an open connected set and $f : U \to \mathbb{C}$ is complex differentiable with $f'(z) = 0$ for each $z \in U$, then there is some constant $w_0 \in \mathbb{C}$ for which $f(z) = w_0$ for all $z \in U$.

Proof: Let $z_0 \in \mathbb{C}$. Then there is some $r > 0$ for which $B_r(z_0) \subset U$. Let r be any positive number for which $B_r(z_0) \subset U$. If $\zeta \in B_r(z_0)$, then there is a straight line path connecting z_0 to ζ in U. Let's try to consider

$$\frac{d}{dt} f((1-t)z_0 + t\zeta).$$

If by this, we just mean

$$\begin{aligned}\frac{d}{dt}\,\mathrm{Re}(f)((1-t)x_0 + t\alpha, (1-t)y_0 + t\beta) + i\,\frac{d}{dt}\,\mathrm{Im}(f)((1-t)x_0 + t\alpha, (1-t)y_0 + t\beta) \\ = \frac{d}{dt} u((1-t)x_0 + t\alpha, (1-t)y_0 + t\beta) + i\,\frac{d}{dt} v((1-t)x_0 + t\alpha, (1-t)y_0 + t\beta),\end{aligned}$$

where $z_0 = x_0 + iy_0$ and $\zeta = \alpha + i\beta$, then we should be able to compute this derivative. In fact, this is what I mean.

Consider the quantity

$$\frac{d}{dt} u((1-t)x_0 + t\alpha, (1-t)y_0 + t\beta).$$

The chain rule from multivariable calculus tells us that if $u \in C^1(\gamma^{-1}(U))$, then

$$\begin{aligned}\frac{d}{dt}&u((1-t)x_0+t\alpha,(1-t)y_0+t\beta)\\ &=\frac{\partial u}{\partial x}((1-t)x_0+t\alpha,(1-t)y_0+t\beta)\ (\alpha-x_0)\\ &\quad+\frac{\partial u}{\partial y}((1-t)x_0+t\alpha,(1-t)y_0+t\beta)\ (\beta-y_0).\end{aligned}$$

We know, however, that $f' = u_x + iv_x$ and $f' \equiv 0$. This means $u_x \equiv 0$, and

$$\frac{\partial u}{\partial x}((1-t)x_0+t\alpha,(1-t)y_0+t\beta)\equiv 0$$

in particular. Similarly, $u_y = -v_x \equiv 0$. Therefore,

$$\frac{d}{dt}u((1-t)x_0+t\alpha,(1-t)y_0+t\beta)\equiv 0$$

as a (continuous) real valued function of one variable. Therefore, by integration and the fundamental theorem of calculus

$$u(\alpha,\beta)=u(x_0,y_0)+\int_0^1\frac{d}{dt}u((1-t)x_0+t\alpha,(1-t)y_0+t\beta)\,dt=u(x_0,y_0).$$

It follows similarly that $\operatorname{Im} f(\zeta) = v(\alpha,\beta) = v(x_0,y_0)$. That is, $f(\zeta) = f(z_0)$, and f is constant on the entire ball $B_r(z_0)$. What we have shown here then is that the set

$$\{z \in U : f(z) = f(z_0)\}$$

is an open subset of $\mathbb{C}$.

Exercise 2.28 Show $\{z \in U : f(z) \neq f(z_0)\}$ is an open subset of $\mathbb{C}$.

According to Exercise 2.28 the connected set U can be expressed as a disjoing union

$$U = \{z \in U : f(z) = f(z_0)\} \cup \{z \in U : f(z) \neq f(z_0)\}$$

of two open sets. Since the first one is not empyty, the second one must be empty and $f(z) \equiv f(z_0)$ is constant. $\square$

Exercise 2.29 Prove the chain rule for composition of a function of two real variables on a differentiable vector valued function: If $u : \mathcal{U} \to \mathbb{R}$ is continuously differentiable on an open set $\mathcal{U} \subset \mathbb{R}^2$, i.e., $u \in C^1(\mathcal{U})$, and $\mathbf{v} : (a, b) \to \mathbb{R}^2$ is a vector valued function defined on an interval $(a, b) \subset \mathbb{R}$, with $\mathbf{v} = (v_1, v_2)$ having component funtions $v_j \in C^1(a, b)$ for $j = 1, 2$, then

$$\frac{d}{dt} u \circ \mathbf{v}(t) = \frac{\partial}{\partial x}(\mathbf{v}(t)) \ \frac{dv_1}{dt}(t) + \frac{\partial}{\partial y}(\mathbf{v}(t)) \ \frac{dv_2}{dt}(t).$$

Aside from what we have covered above, there is some other terminology of minor importance. We know there is a definition of what it means for a function $f : U \to \mathbb{C}$ to be complex differentiable **at a point**. We are primarily interested in situations in which $f : U \to \mathbb{C}$ is complex differentiable at every point in an open set U. In this case, the function f is variously called complex differentiable, **analytic**, **holomorphic**, and maybe even regular—though the last one is not used too much any more. The examples in § 26 of BC are good to consider. Please consider them.

Harmonic functions

The big assuption here is that if $f = u + iv : U \to \mathbb{C}$ is complex differentiable, then $u, v \in C^2(\gamma^{-1}(U))$. Once you know that, showing the real and imaginary parts u and v satisfy Laplace's equation is straightforward. This is Problem 10 is Assignment 6.

If you haven't noted it already, it's good to be aware of what Laplace's equation looks like, and it's good to know that solutions of Laplace's equation are called **harmonic functions**.

The fact that these functions $u = \text{Re}(f)$ and $v = \text{Im}(f)$, which nominally only need the first order partial derivatives to exist in order to satisfy the conditions for f' to exist or to satisfy the Cauchy-Riemann equations, actually turn out to have so many unexpected partial derivatives is called **regularity** or **additional regularity**.

The level sets of harmonic functions are interesting to consider as is done briefly in BC in the exercises at the end of § 27. I suggest we try to come back to this.

Analytic continuation and the reflection principle

I discussed these briefly above, and I'll just mention here that they are discussed in the last two sections of Chapter 2 in BC; we should come back to them after Chatper 4 in BC (or maybe after Chatper 5).

Chapter 3

Elementary Functions

These notes are from Chapter 3 of BC.

Let me recall that Chapter 2 was primarily about the general notion of what it means for a **complex function** $f : \Omega \to \mathbb{C}$ where Ω is an open subset of $\mathbb{C}$ to be (complex) **differentiable**.

At this point I think/hope you are fairly familiar with the complex differentiable function $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$. In fact, you should hopefully be fairly familiar with $f : \mathbb{C} \to \Sigma = \Sigma_1 \cup \Sigma_2$ where Σ is a/the **Riemann surface** associated with $f(z) = z^2$ consisting of two sheets and having **branch cuts** and a **branch point** (at $z = 0$). The square function, then, has also a well-defined inverse $f^{-1} : \Sigma \to \mathbb{C}$ which we may denote by $f^{-1}(z) = \sqrt{z}$, but technically (and practically) this function will be considered and defined in terms of **branches of the square root**.

Though we will not discuss it further here, we have also discussed to some extent the values of $f(z) = z^n$, the associated Riemann surfaces and inverses, that is taking n-th roots when n is a natural number.

We have also discussed, to a certain extent, the **complex exponential function** $f(z) = e^z$, though if we consider the situation honestly, we have really only considered the **restriction** of this function to the real axis where

$$f(x) = e^x$$

is the familiar real exponential and to the imaginary axis where

$$f(iy) = e^{iy} = \cos y + i \ \sin y$$

is a function which behaves quite differently. Probably it's fair to say, that we, i.e., you, didn't actually think about $f(iy)$ in too much detail. It is now time to do that, and furthermore to extend the same consideration to the full complex exponential

$$f(z) = e^x(\cos y + i\ \sin y)$$

as well as various other functions, perhaps quite notably the **complex trigonometric functions**.

In summary, there are quite a few complex functions with which one should become familiar. These include

1. integer power functions,
2. polynomials,
3. various branches of roots,
4. the exponential function and its inverse the complex logarithm,
5. more general powers both of the variable $f(z) = z^\alpha$ and with variable power $f(z) = \alpha^z$,
6. trigonometric functions, $\sin z$, $\cos z$, $\tan z$, $\csc z$, etc. and their inverses,
7. and also hyperbolic functions like $\cosh z$ and $\sinh z$, which turn out to be nominally simpler functions simply related to the complex exponential, but are probably somewhat less familiar to you even in their real forms.

I think that's a pretty good list. You probably won't become an expert on all of these functions this semester, but hopefully you can master the use and understanding of some of them, and this will put you in a position to deal with the others when the need arises. All of them can (and I might say probably should) be thought of in terms of mappings. There is usually a Riemann surface involved, and it helps to identify **fundamental domains** which is something I will try to discuss below. Perhaps that is enough of an introduction, so let's try to get started with §30 of BC.

3.1 The complex exponential

As is mentioned in BC, one way to **define** the complex exponential $\exp : \mathbb{C} \to \mathbb{C}$ is by taking the formula above, which in turn is based on Euler's formula:

$$\exp(z) = e^x(\cos y + i\ \sin y). \tag{3.1}$$

It may be mentioned that there are other ways to define the complex exponential, most notably in terms of power series as

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\, z^n.$$

Then one can derive Euler's formula and (3.1) as consequences, but that is not the point of view we are going to take here. This important approach is taken up in Chapter 5 of BC. Let's begin by considering the Euler formula as a mapping. First of all, we might note that horizontal lines $L = \{z \in \mathbb{C} : \mathrm{Im}(z) = y_0\}$ map (in)to rays

$$\{e^x e^{iy_0} : x \in \mathbb{R}\}$$

with an endpoint at $w = 0$ and having fixed argument y_0. Each such ray intersects the unit circle at a point corresponding to $\mathrm{Re}(z) = x = 0$ at $w = e^{iy_0}$. The portion

$$\{x + iy_0 : x < 0\}$$

of the line L, which is a half-line, maps into the unit disk between $w = 0$ and $w = e^{iy_0}$. The right half line $\{x + iy_0 : x > 0\}$ maps to a ray extending from $w = e^{iy_0}$ to $w = \infty$ in the Riemann sphere.

The horizontal strip $\Sigma_0 = \{z = x + iy : 0 \le y < 2\pi\}$, which of course consists of a particular union of horizontal lines is called a **fundamental domain** for e^z and has image $\mathbb{C}\backslash\{0\}$. We can call this particular image $\mathcal{L}_0$ and note that it is one sheet of the Riemann surface associated with the exponential function. It should be more or less clear that $f(z) = \exp(z)$ restricted to a fundamental domain Σ_0 is a bijection onto its image. The inverse $\log_0 : \mathcal{L}_0 \to \Sigma_0$ is a branch of the complex logarithm. Notice $z = \log_0(w)$ is the complex number for which

$$e^z = e^x(\cos y + i \sin y) = w \qquad \text{and} \qquad 0 \le y < 2\pi.$$

Taking the modulus of both sides in the relation $e^z = e^x(\cos y + i \sin y) = w$, we find

$$e^x = |w|.$$

Consequently, we can find the real part of $\log_0(w)$ using the familiar real logarithm $\ln : (0, \infty) \to \mathbb{R}$. That is,

$$\mathrm{Re}\log_0(w) = x = \ln|w| = \ln\sqrt{\mathrm{Re}(w)^2 + \mathrm{Im}(w)^2}. \tag{3.2}$$

Exercise 3.1 (complex exponential)

(a) It may occur to you that while

$$e^z = e^x(\cos y + i \sin y)$$

gives a well-defined formula for the function $\exp : \mathbb{C} \to \mathbb{C}\backslash\{0\}$ or alternatively

$$\exp : \mathbb{C} \to \mathcal{L} = \bigcup_{j=-\infty}^{\infty} \mathcal{L}_j,$$

it is not immediately clear that this function is complex differentiable. Use the sufficient condition associated with the Cauchy-Riemann equations to show exp is complex differentiable.

(b) Use the chain rule to show the branch $\log_0 : \mathcal{L}_0 \to \Sigma_0$ defined above is complex differentiable (at least on $\text{int}(\mathcal{L}_0)$).

(c) Recall that the real and imaginary parts of a complex differentiable function are harmonic. What interesting harmonic function do you see in (3.2)?

The imaginary part of $\log_0(w)$ is a little more complicated to contemplate. Indeed, we know there is a unique $y \in [0, 2\pi)$ for which $\cos y + i \sin y = w/|w|$, but there is not a simple formula. We have encountered this value before, however, and we have a nice notation:

$$\log_0(w) = \ln |w| + i \operatorname{Arg}(w).$$

How about this?

$$\operatorname{Im} \log_0(w) = \begin{cases} \tan^{-1}(y/x), & x > 0, y \geq 0, \text{(first quadrant)} \\ \cot^{-1}(x/y), & y > 0, \text{(upper half plane)} \\ \tan^{-1}(y/x) + \pi, & x < 0, \text{(left half plane)} \\ \cot^{-1}(x/y) + \pi, & y < 0, \text{(lower half plane)} \\ \tan^{-1}(y/x) + 2\pi, & x > 0, y \leq 0, \text{(fourth quadrant).} \end{cases}$$

Here I have used what are called the principal real branches of arctangent and arccotangent which are the inverses of $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ and $\cot : (0, \pi) \to \mathbb{R}$ respectively.

Hopefully, it's clear to you that these nice smooth principal real branches of arctangent and arccotangent can be used to find a nice non-singular formula for the complex argument, and consequently for the imaginary part of the complex logarithm, locally near any point in any specified sheet of $\mathcal{L}$. Roughly speaking then, we can write

$$\log w = \ln |w| + i \arg(w),$$

though we (may) have to be a little careful with the imaginary part here.

The complex exponential and complex logarithm are rather important functions. I will temporarily finish these notes on this topic with a final observation: The complex exponential is **complex periodic**, meaning there is a complex number $\omega \in \mathbb{C}\backslash\{0\}$ for which

$$\exp(z + \omega) = \exp(z) \qquad \text{for all } z \in \mathbb{C}.$$

The complex period of the exponential function is $\omega = 2\pi i$.

Exercise 3.2 (complex periodicity) Define what it means for a complex differentiable function $f : U \to \mathbb{C}$ defined on an open subset U of $\mathbb{C}$ to have complex period ω. Be careful to state any conditions that must be satisfied by the domain U.

Exercise 3.3 (complex periodicity) Find an entire function with period $1 + i$.

3.2 Complex Powers

These are notes starting in § 35 of BC.

Given a branch log of the complex logarithm and a complex number $\alpha \in \mathbb{C}$, a branch of z^α is defined by

$$z^\alpha = e^{\alpha \log z}.$$

Exercise 3.4 (complex powers, Exercises 3.36.1-3 in BC) Discuss the values of

(a) $(1 + i)^i$.

(b) $(-i)^i$.

(c) $(-1 + \sqrt{3}\, i)^{3/2}$.

Given a branch log of the complex logarithm and a complex number $\alpha \in \mathbb{C}$, a branch of α^z is defined by

$$\alpha^z = e^{z \log \alpha}.$$

Exercise 3.5 (complex powers) How does the exponential function e^z compare/correspond to the branches of

$$e^{z \log e}?$$

3.3 Trigonometric functions

Notes starting in § 37 of BC.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Exercise 3.6 Show the complex cosine and complex sine are entire.

Exercise 3.7 Compute

$$\frac{d}{dz} e^{iz}.$$

Exercise 3.8 (mapping properties) Find a fundamental domain for the complex cosine, i.e., a subset E of $\mathbb{C}$ for which the image

$$\{\cos(z) : z \in E\} = \mathbb{C}.$$

Exercise 3.9 (mapping properties) Find a fundamental domain for the complex sine, i.e., a subset E of $\mathbb{C}$ for which the image

$$\{\sin(z) : z \in E\} = \mathbb{C}.$$

Exercise 3.10 (formula) Find formulas for the real and imaginary parts of $\cos z$ as functions of x and y in $z = x + iy$.

Exercise 3.11 (formula) Find formulas for the real and imaginary parts of $\sin z$ as functions of x and y in $z = x + iy$.

When you finish the exercises above you should be able to see the complex cosine and sine as functions with fundamental domains given by strips. Each strip corresponds to a sheet in the Riemann surface for the function, and the sheets have two branch points at ± 1 with cuts extending to $w = \infty$ along the real axis and a non-singular segment $[-1, 1]$ on the real axis.

For each sheet in the Riemann surface there is a branch of the inverse. For example, the cosine has fundamental domains

$$A_j = \{z \in \mathbb{C} : j\pi < \text{Re}(z)(j+1)\pi\} \cup \{j\pi + iy : y \le 0\} \cup \{(j+1)\pi + iy : y > 0\}$$

for $j \in \mathbb{Z}$. These correspond to sheets $\mathcal{C}_j$ for $j \in \mathbb{Z}$ and associated to each sheet is a branch of $\cos^{-1} = \text{arccos}_j : \mathcal{C}_j \to A_j$ such that

$$\cos \circ \text{arccos}_j(w) = w \in \mathcal{C}_j.$$

It takes a little checking, but some version of the usual derivative formulas

$$\frac{d}{dw}\operatorname{arccos}_j(w) = -\frac{1}{\sqrt{1-x^2}} \qquad \text{and} \qquad \frac{d}{dw}\operatorname{arcsin}_j(w) = \frac{1}{\sqrt{1-x^2}}$$

also hold. The point one has to be careful about is specifying the particular branch of the square complex square root that should be used.

3.4 Complex tangent

Maybe it can be said that we need to up our game a bit to deal with the complex tangent, but basically this function is not so different. One significant difference is that branch points in the Riemann surfaces considered so far have been the images of points in the finite part of the Riemann sphere or is not included in the Riemann surface. For example, the branch point for $f(z) = z^n$ is at $w = 0 = f(0)$. The branch point for $f(z) = e^z$ is at $w = 0$. There is of course no z for which $e^z = 0$. Finally, it will be recalled that the branch points for $f(z) = \cos z$ are located at $z = \pm 1$ with $-1 = \cos\pi$ and $1 = \cos 0$.

Let me just say that something new happens in this regard with respect to tangent, so keep an eye out for it.

I guess we should start with a formula for the real and imaginary parts. It will help to note/recall some double angle formulas:

$$\cos 2x = \cos^2 x - \sin^2 x \qquad \text{and} \qquad \sin 2x = 2\cos x \sin x.$$

Also,

$$\cosh 2y = \cosh^2 y + \sinh^2 y \qquad \text{and} \qquad \sinh 2y = 2\cosh y \sinh y.$$

These latter formulas are consequences of the formulas

$$\cosh^2 y = \frac{e^{2y} + 2 + e^{-2y}}{4} \qquad \text{and} \qquad \sinh^2 y = \frac{e^{2y} - 2 + e^{-2y}}{4}$$

for the squares of

$$\cosh y = \frac{e^y + e^{-y}}{2} \qquad \text{and} \qquad \sinh y = \frac{e^y - e^{-y}}{2}.$$

One also has

$$\cos^2 x + \sin^2 x = 1 \qquad \text{and} \qquad \cosh^2 y - \sinh^2 y = 1.$$

Finally, we recall the important formulas we needed for the complex cosine and sine:

$$\cos(x+iy) = \cos x \cosh y - i\ \sin x \sinh y \qquad \text{and} \qquad \sin(x+iy) = \sin x \cosh y + i\ \cos x \sinh y.$$

With these formualas in mind we turn to $\tan z = \tan(x+iy)$.

$$\begin{aligned}
\tan z &= \frac{\sin x \cosh y + i\ \cos x \sinh y}{\cos x \cosh y - i\ \sin x \sinh y}\,\frac{\cos x \cosh y + i\ \sin x \sinh y}{\cos x \cosh y + i\ \sin x \sinh y} \\
&= \frac{\cos x \sin x(\cosh^2 y - \sinh^2 y) + i\ \cosh y \sinh y(\cos^2 x + \sin^2 x)}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y} \\
&= \frac{\cos x \sin x}{\cos^2 x + \sinh^2 y} + i\ \frac{\cosh y \sinh y}{\cos^2 x + \sinh^2 y} \\
&= \frac{\sin 2x}{\cos 2x + \cosh 2y} + i\ \frac{\sinh 2y}{\cos 2x + \cosh 2y}
\end{aligned}$$

because

$$\begin{aligned}
2\cos^2 x + 2\sinh^2 y &= \cos^2 x + 1 - \sin^2 x + \sinh^2 y + \cosh^2 y - 1 \\
&= \cos 2x + \cosh 2y.
\end{aligned}$$

The real interval $(-\pi/2, \pi/2)$ maps to the entire real axis monotonically with $\tan(0) = 0$. We know this from the real tangent. Setting $x = 0$ in the main formula

$$\tan(x+iy) = \frac{\sin 2x}{\cos 2x + \cosh 2y} + i\ \frac{\sinh 2y}{\cos 2x + \cosh 2y} \tag{3.3}$$

we see

$$\tan iy = i\frac{2\cosh y \sinh y}{1 + \cosh^2 y + \sinh^2 y} = i\frac{\sinh y}{\cosh y} = i \tanh y.$$

Also,

$$\tan\left(\frac{\pi}{2} + iy\right) = i\frac{2\cosh y \sinh y}{-1 + \cosh^2 y + \sinh^2 y} = i\frac{\cosh y}{\sinh y} = i \coth y.$$

The real functions $\tanh y$ and $\coth y$ may not be entirely familiar. The real hyperbolic tangent is an extremely nice function. It has no singularity. It is increasing and bounded. Its values are asymptotic to 1 on the right and to -1 on the left much like $2\tan^{-1}(x)/\pi$ and it has derivative $\operatorname{sech}^2(0) = 1$ at $x = 0$ like $\tan^{-1}(x)$. See Figure 3.1. The real hyperbolic cotangent, on the other hand, does have a singularity at $x = 0$. All values of this function are greater than one, and the values tend to 1 as $|x| \to \infty$.

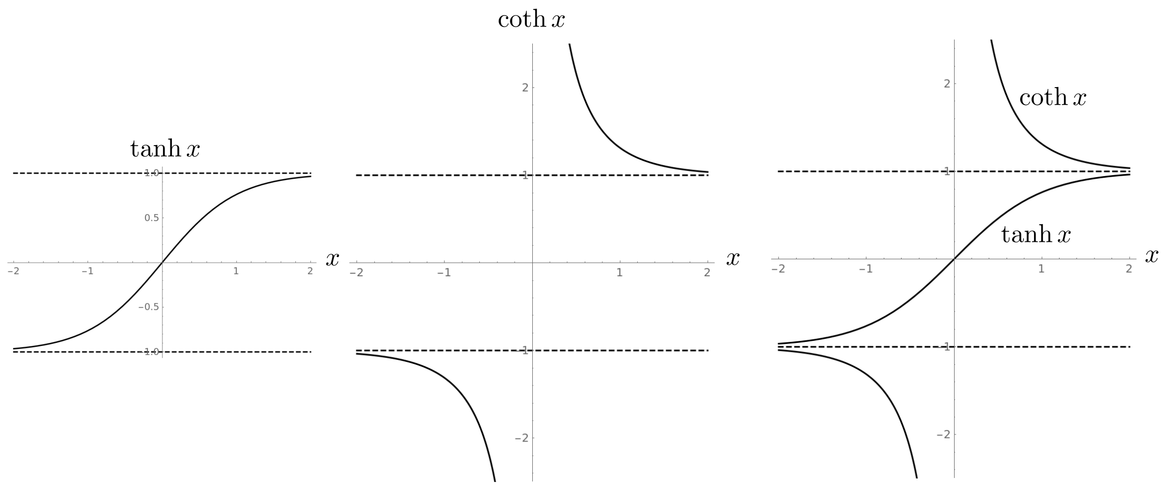

Figure 3.1: The real hyperbolic tangent and cotangent.

The function is smoothly decreasing on each of the individual intervals $x < 0$ and $x > 0$.

Having made these observations, let us turn to the properties of $\tan z$ as a map of the complex plane into itself, or more properly into an appropriate Riemann surface for *tanz*. The familiar singular behavior of the real tangent, $\tan x$ tells us that the real interval between $z = x = -\pi/2$ and $z = x = \pi/2$ maps to the entire real line, symmetrically with $\tan(0) = 0$. The entire vertical also, in view of the relation $\tan(iy) = i \tanh y$ on the other hand, maps monotonically and symmetrically onto the finite open interval between $-i$ and i. These mapping properties of the complex tangent on the axes are indicated in Figure 3.2.

Notice the limit *iyto*∞ with y positive and increasing in the domain correspond to the limit

$$\lim_{y \nearrow \infty} \tan(iy) = i.$$

In fact, this is an increasing limit, so we can informally include some additional information by writing

$$\tan(iy) \nearrow i \qquad \text{as} \qquad y \nearrow \infty,$$

though this is in no way intended to imply an ordering among complex numbers. There is an ordering on the imaginary axis. With this in mind, consider the line

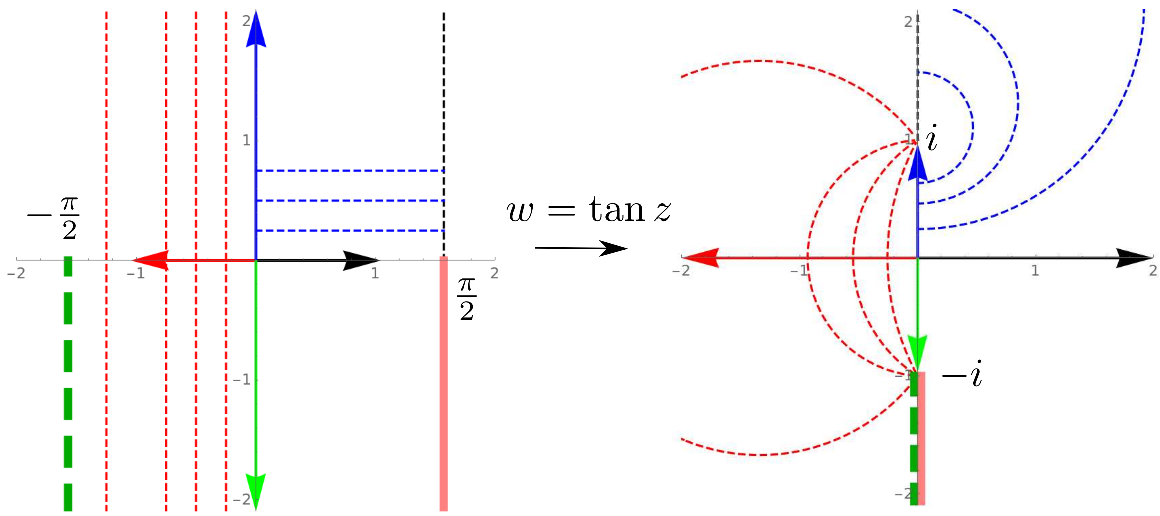

Figure 3.2: The complex tangent on a fundamental domain.

$\operatorname{Re} z = \pi/2$ and the limit $\pi/2 + iy$ as $y \nearrow \infty$ in particular. According to the formula $\tan(\pi/2 + iy) = i \coth y$, we see

$$\tan(\pi/2 + iy) \searrow i \qquad \text{as} \qquad y \nearrow \infty.$$

This is also indicated by the black dashed line in Figure 3.2. More generally, we can consider the vertical half lines $\operatorname{Re}(z) = x + iy$ with $0 < x < \pi/2$ and the limits

$$\lim_{y \nearrow \infty} \tan(x + iy)$$

in particular. It is easy to see from (3.3) that this limit has value i as well. Thus, for each fixed x with $-\pi/2 < x < \pi/2$ the image

$$\{\tan(x + iy) : y > 0\}$$

is a curve in the first quadrant meeting the real axis at (a right angle at) $w = \tan x$ and connecting this point to $w = i$. These image curves are not drawn in Figure 3.2, but some similar images of vertical lines $\operatorname{Re}(z) = x$ with $-\pi/2 < x < 0$ are drawn with red dashed lines. The graphic suggests each such image curve is a circle, or portion of a circle, with center and radius depending only on x. The actual location of the center and value of the radius are not obvious from the geometry, so let's set aside the suggestion for a moment and consider instead a different collection of images.

The image of each horizontal segment $\{x + iy : 0 \leq x \leq \pi/2\}$ for $y > 0$ fixed coresponding to the blue dashed lines in Figure 3.2 also appears to be a cirlce. In this case, we know the images of the endpoints:

$$\tan(iy) = i \tanh y \qquad \text{and} \qquad \tan(\pi/2 + iy) = i \coth y.$$

Thus, if the image is a circle meeting the imaginary axis at a right angle as suggested by the graphic, then the center and radius of this circle should be given by

$$w = \frac{i}{2}(\tanh y + \coth y) = i\ \frac{\sinh^2 y + \cosh^2 y}{2\cosh y \sinh y} = i\ \frac{\cosh 2y}{\sinh 2y} = i\ \coth 2y$$

and

$$r = \frac{1}{2}(\coth y - \tanh y) = \frac{\cosh^2 y - \sinh^2 y}{2\cosh y \sinh y} = \frac{1}{\sinh 2y}$$

respectively. That the image is actually a semicircle with this center and radius can be verified as follows:

$$
\begin{aligned}
|\tan(x+iy)-w|^2 &= \left(\frac{\sin 2x}{\cos 2x+\cosh 2y}\right)^2 + \left(\frac{\sinh 2y}{\cos 2x+\cosh 2y} - \coth 2y\right)^2 \\
&= \frac{\sin^2 2x \sinh^2 2y + [\sinh^2 2y - \cosh 2y(\cos 2x+\cosh 2y)]^2}{\sinh^2 2y(\cos 2x+\cosh 2y)^2} \\
&= \frac{1}{\sinh^2 2y(\cos 2x+\cosh 2y)^2}\Big\{ \\
&\qquad \sinh^2 2y[\sin^2 2x + \sinh^2 2y - 2\cosh 2y(\cos 2x+\cosh 2y)] \\
&\qquad\qquad + \cosh^2 2y(\cos 2x+\cosh 2y)^2\Big\} \\
&= \frac{1}{\sinh^2 2y(\cos 2x+\cosh 2y)^2}\Big\{ \\
&\qquad \sinh^2 2y[-\cos^2 2x + \cosh^2 2y - 2\cosh 2y(\cos 2x+\cosh 2y)] \\
&\qquad\qquad + \cosh^2 2y(\cos 2x+\cosh 2y)^2\Big\} \\
&= \frac{1}{\sinh^2 2y(\cos 2x+\cosh 2y)^2}\Big\{ \\
&\qquad \sinh^2 2y(\cos 2x+\cosh 2y)[\cosh 2y - \cos 2x - 2\cosh 2y] \\
&\qquad\qquad + \cosh^2 2y(\cos 2x+\cosh 2y)^2\Big\} \\
&= \frac{\sinh^2 2y(-1)+\cosh^2 2y}{\sinh^2 2y} \\
&= \frac{1}{\sinh^2 2y}.
\end{aligned}
$$

The images of the vertical lines are also circles, though as mentioned above, the centers and radii of these circles may not be immediately obvious. Upon further reflection on the following two suggestions

(i) The complementary circular arc of

$$\{\tan(x+iy) : y \in \mathbb{R}\},$$

which we assume to be part of a circle with center on the real axis, must be the image of a vertical line $\operatorname{Re} z = \xi$ with $\xi < 0$ taking a value “symmetric” (in some sense) to the value $x > 0$.

(ii) It is possible (or even likely) that the intersection points $\xi < 0 < x$ and the corresponding center w and radius r have values (in some sense) algebraically symmetric with the values

$$\tan(iy) = i\tanh y, \qquad \tan(\pi/2+iy) = i\coth y, \qquad w = i\coth 2y, \qquad \text{and} \qquad r = \operatorname{sech} 2y$$

for the circles passing through the imaginary axis.

A first value to try is $\xi = x - \pi/2$. This gives

$$\tan x \qquad \text{and} \qquad \tan\left(\frac{\pi}{2} - x\right) = -\coth x.$$

It will be noted that these values do indeed bear some resemblance to the corresponding image values for the horizontal lines, and thus $\xi = x - \pi/2$ is a natural value to consider "symmetric" to x. Indeed, assuming a circle is orthogonal to the real axis and passes through these values, we have for the center and radius

$$w = \frac{1}{2}(\tan x - \cot x) = -\frac{\cos^2 x - \sin^2 x}{2\cos x \sin x} = -\cot 2x$$

and

$$r = \frac{1}{2}(\cot x + \tan x) = \frac{\cos^2 x + \sin^2 x}{2\cos x \sin x} = \frac{1}{\sin 2x}.$$

By this time, we should start to believe we are on the right track. Indeed a calculation much like that for the images of the horizontal segments confirms out suspicion.

$$
\begin{aligned}
|\tan(x+iy)-w|^2 &= \left(\frac{\sin 2x}{\cos 2x+\cosh 2y}+\cot 2x\right)^2+\left(\frac{\sinh 2y}{\cos 2x+\cosh 2y}\right)^2\\
&= \frac{[\sin^2 2x+\cos 2x(\cos 2x+\cosh 2y)]^2+\sin^2 2x\sinh^2 2y}{\sin^2 2x(\cos 2x+\cosh 2y)^2}\\
&= \frac{1}{\sin^2 2x(\cos 2x+\cosh 2y)^2}\Big\{\\
&\qquad \sin^2 2x[\sin^2 2x+\sinh^2 2y+2\cos 2x(\cos 2x+\cosh 2y)]\\
&\qquad\qquad +\cos^2 2x(\cos 2x+\cosh 2y)^2\Big\}\\
&= \frac{1}{\sin^2 2x(\cos 2x+\cosh 2y)^2}\Big\{\\
&\qquad \sin^2 2x[-\cos^2 2x+\cosh^2 2y+2\cos 2x(\cos 2x+\cosh 2y)]\\
&\qquad\qquad +\cos^2 2x(\cos 2x+\cosh 2y)^2\Big\}\\
&= \frac{1}{\sin^2 2x(\cos 2x+\cosh 2y)^2}\Big\{\\
&\qquad \sin^2 2x(\cos 2x+\cosh 2y)[\cosh 2y-\cos 2x+2\cos 2x]\\
&\qquad\qquad +\cos^2 2x(\cos 2x+\cosh 2y)^2\Big\}\\
&= \frac{\sin^2 2x+\cos^2 2x}{\sin^2 2x}\\
&= \frac{1}{\sin^2 2x}.
\end{aligned}
$$

3.5 Derivatives

We could (and perhaps should) give a little more consideration of formulas like

$$\frac{d}{dw}\sin^{-1} w=\frac{1}{\sqrt{1-w^2}} \qquad \text{and} \qquad \frac{d}{dw}\tan^{-1} w=\frac{1}{1+w^2}.$$

The point is, above all, that these formulas involve branches of the various functions involved, and those branches should be specified carefully and the values checked/considered.

3.6 Summary

We've discussed some elementary functions, starting with the complex square function $f(z) = z^2$. Hopefully, you understand this function and its Riemann surface and the branches of its inverse completely. The integer power functions $f(z) = z^n$ for $n = 3, 4, 5, \ldots$ have n-sheeted Riemann surfaces and n branches of inverse functions, but they are rather direct generalizations of $f(z) = z^2$.

We've also discussed the complex exponential function $f(z) = e^z$ and its Riemann surface with countably many sheets each of which is a domain for a branch of the complex logarithm $\log(w)$. The branch point $w = 0$ itself is not included becuase $e^z \neq 0$ for any $z \in \mathbb{C}$.

Finally, we have the complex trigonometric functions $\cos z$, $\sin z$ and $\tan z$. Each of these have two branch points in each sheet of the Riemann surface, though the tangent has a rather different character as a mapping. In particular, the complex cosine and complex sine are completely regular, with the branch points $-1 = \cos(\pi)$, $1 = \cos(0)$, and $\pm 1 = \sin(\pm\pi/2)$ regular points in the respective Riemann surfaces while the complex tangent has both a sequence of countably many (double) poles at $\pi/2 + k\pi$ for $k \in \mathbb{Z}$ and excluded branch points at $\pm i$ with $\tan z \neq \pm i$ for any $z \in \mathbb{C}$. Again, there are the naturally associated branches of arccosine, arcsine, and arctangent.

The complex exponential has an essential singularity at $z = \infty$. We will discuss this topic more later. The same thing can be said about the complex cosine and complex sine.

This is a good start. There are many other interesting and important complex functions, but if you understand how to deal with the ones listed here, you are in a good position to at least do a basic analysis of functions like the complex Gamma function, the Lambert W function, the Riemann zeta function, and most any other complex function you might run into (or at least know what such an analysis looks like). To actually complete the analysis for many of these functions it is helpful to know a bit more complex analysis and something about complex integration in particular.

Chapter 4

Complex Integration

4.1 Complex valued functions of one real variable

This section comprises my notes for sections 41 and 42 of BC.

I really like the way Brown and Churchill do this. Let me emphasize that these functions $\gamma : (a, b) \to \mathbb{C}$ with

$$\gamma(t) = \alpha(t) + i\ \beta(t)$$

and $(a, b) \subset \mathbb{R}$, which Brown and Scriven call[1] $w = w(t)$, and are **not** complex functions $f : U \to \mathbb{C}$ of a complex variable $z \in U \subset \mathbb{C}$ as we've been discussing up until now. These functions are something different. They are very different from $f : U \to \mathbb{C}$, but they are not very different from something else (or some other things) you know.

Primarily it should be noted that a complex valued function $\gamma = \alpha + i\ \beta$ has a **derivative** and an **integral** which are given by the **definitions**

$$\frac{d\gamma}{dt} = \frac{d\alpha}{dt} + i\ \frac{d\beta}{dt} \tag{4.1}$$

and

$$\int_a^b \gamma(t)\, dt = \int_a^b \alpha(t)\, dt + i\ \int_a^b \beta(t)\, dt \tag{4.2}$$

[1]Brown and Churchill also write $w(t) = u(t) + i\ v(t)$ using the same symbols for the real and imaginary parts as is customarily used for the real and imaginary parts $u = u(x, y)$ and $v = v(x, y)$ where $z = x+iy$ of a real complex valued function of a complex variable. I think for an introduction, it may be better to emphasize the distinction a little more explicity in the notation, so I've chosen $\gamma = \alpha + i\ \beta$.

when α and β are differentiable on (a, b) and continuous[2] on $[a, b]$ respectively. These derivatives and integrals have certain properties like linearity

$$(c_1\gamma + c_2\tilde{\gamma})' = c_1\gamma' + c_2\tilde{\gamma}' \qquad \text{and} \qquad \int (c_1\gamma + c_2\tilde{\gamma}) = c_1 \int \gamma + c_2 \int \tilde{\gamma},$$

but it is probably fair to say that these properties are not different in any significant way from the properties of derivatives and integrals of real valued functions of one real variable with which you are familiar from Calculus 1. In particular, the functions I have denoted by α and β are just real valued functions of one real variable like the functions considered in Calculus 1.

Again, I want to emphasize that the derivative defined in (4.1) is not a complex derivative like we have discussed in Chapter 2. Furthermore, we are going to define a **complex integral** which looks something like

$$\int_\Gamma f$$

associated with a complex valued function $f : U \to \mathbb{C}$ of a complex variable $z \in U$, and this is/will be a really new thing. We will use integrals of a complex valued function of a real variable defined in (4.2) simply as a tool to talk about real complex integrals. Of course, you still should get comfortable with using integrals and derivatives like the ones defined in (4.2) and (4.1).

Exercise 4.1 Let $\gamma = \alpha + \beta\, i$ be a complex valued function of one real variable t. Verify the following:

(a) (Exercise 4.42.1 in BC) $(c\gamma)' = c\gamma'$ where $c = a + bi \in \mathbb{C}$.

(b) $(f \circ \gamma)' = f' \circ \gamma\ \gamma'$ where $f : U \to \mathbb{C}$ is complex differentiable on an open set $U \subset \mathbb{C}$ and $\gamma : (a, b) \to U$.

(c)

$$\int_a^b \gamma'(t)\, dt = \gamma(b) - \gamma(a).$$

(d)

$$\frac{d}{dt} \int_a^t \gamma(\tau)\, d\tau = \gamma(t).$$

[2] Technically, I should say α and β extend continuously to the closed interval $[a, b]$ here. Alternatively, I could integrate on some subinterval $[t_1, t_2] \subset (a, b)$, but this involved more symbols than I wanted to use.

Some things are, believe it or not, simpler when these integrals are used. For example, you may remember that integrals like

$$\int_0^\pi e^x \cos x\, dx$$

normally require integration by parts (and maybe even repeated integration by parts) to compute. However, look at this:

$$\begin{aligned}
\int_0^\pi e^{(1+i)t}\, dt &= \frac{1}{1+i} e^{(1+i)t} \Big|_{t=0}^{\pi} \\
&= \frac{1-i}{2}\left[e^{(1+i)\pi} - 1\right] \\
&= \left(\frac{1}{2} - \frac{i}{2}\right)\left[-e^\pi - 1\right] \\
&= -\frac{e^\pi + 1}{2} + \frac{e^\pi + 1}{2}\, i.
\end{aligned}$$

On the other hand,

$$e^{(1+i)t} = e^t(\cos t + i \sin t),$$

so

$$\int_0^\pi e^{(1+i)t}\, dt = \int_0^\pi e^t \cos t\, dt + i \int_0^\pi e^t \sin t\, dt.$$

Equating real and imaginary parts, we get

$$\int_0^\pi e^t \cos t\, dt = -\frac{e^\pi + 1}{2} \qquad \text{and} \qquad \int_0^\pi e^t \sin t\, dt = \frac{e^\pi + 1}{2}$$

without any explicit use of integration by parts. (Cool!)

4.2 Complex Integration

As mentioned above a complex integral

$$\int_\Gamma f$$

of a function $f : U \to \mathbb{C}$ is an integral over a **curve** $\Gamma \subset U$. A curve can (essentially always) be **parameterized** by a function

$$\alpha : [a, b] \to U$$

with $\alpha = \alpha_1 + i\alpha_2$ **continuous**, meaning the real and imaginary parts $\alpha_1 = \operatorname{Re}\alpha$ and $\alpha_2 = \operatorname{Im}\alpha$ are continuous real valued functions on the interval $[a, b]$. Most times we will integrate over curves Γ which can be parameterized with α_1 and α_2 continuously differentiable (except perhaps at isolated points). Some details about curves will/should become clear as we go along. Here is something good to know about curves:

> If $\alpha = \alpha_1 + i\alpha_2 : [a, b] \to U$ is continuously differentiable, then the **arclength** along Γ can be defined by
>
> $$s = \int_a^t |\alpha'(\tau)|\, d\tau.$$
>
> In this case, s is an non-decreasing, continuously differentiable function of t with
>
> $$\frac{ds}{dt} = |\alpha'(t)|.$$
>
> If $|\alpha'(t)| \neq 0$, then s is increasing with an **inverse** $\xi : [0, L] \to [a, b]$ where
>
> $$L = \int_a^b |\alpha'(\tau)|\, d\tau$$
>
> is the **total length** of Γ. The inverse is also differentiable and
>
> $$\xi'(s) = \frac{1}{|\alpha' \circ \xi(s)|}.$$
>
> Furthermore, Γ can be **reparameterized by arclength**. Specifically, we can take $\gamma : [0, L] \to U$ by
>
> $$\gamma(s) = \alpha \circ \xi(s)$$
>
> and then
>
> $$\dot{\gamma} = \frac{d\gamma}{ds} = \frac{\alpha' \circ \xi(s)}{|\alpha' \circ \xi(s)|}$$
>
> has unit modulus.

Exercise 4.2 Parameterize $\partial B_r(z_0)$ by argument θ based at z_0 and reparameterize by arclength.

Exercise 4.3 Parameterize the straight line segment from $z_0 = x_0 + iy_0$ to $z_1 = x_1 + iy_1$ with $\alpha(j) = z_j$ for $j = 0, 1$. Reparameterize by arclength.

4.3 Starter Definition

$$\int_\Gamma f = \int_a^b f \circ \alpha(t)\ \alpha'(t)\, dt \tag{4.3}$$

where α is a continuously differentiable parameterization of $\Gamma \subset U$ and $f : U \to \mathbb{C}$ is a complex valued function defined on an open set $U \subset \mathbb{C}$ as usual. Here we do not need f to be complex differentiable, but usually we will have that. We do need f to be continuous.

Technically we need to show the right side of (4.3) does not depend on the particular parameterization we use for Γ, but let's ignore that detail for now. Here are some interesting complex integrals:

$$\begin{aligned}\int_{\partial B_r(z_0)} \frac{1}{z - z_0} &= \int_0^{2\pi} \frac{1}{z_0 + re^{it} - z_0}\ ire^{it}\, dt\\ &= i\int_0^{2\pi} 1\, dt\\ &= 2\pi.\end{aligned}$$

$$\begin{aligned}\int_\Gamma z &= \int_a^b \gamma(t)\ \gamma'(t)\, dt\\ &= \int_a^b \frac{d}{dt}\left[\frac{1}{2}\gamma(t)^2\right] dt\\ &= \frac{1}{2}\gamma(t)^2\Big|_a^b\\ &= \frac{1}{2}[\gamma(b)^2 - \gamma(a)^2].\end{aligned} \tag{4.4}$$

Exercise 4.4 Generalize (4.3) by allowing a **piecewise continuously differentiable** parameterization of Γ. That is, consider $\alpha : [a, b] \to U$ a continuous parameterization with

$$\alpha\Big|_{[t_{j-1}, t_j]} \quad \text{continuously differentiable for } j = 1, 2, \ldots, n$$

where $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ is a partition of the interval $[a, b]$.

Exercise 4.5 Show (4.4) still gives a valid formula for $\int_\Gamma z$ for integrals of the type considered in Exercise 4.4.

4.4 Integration on Riemann surfaces

Section 46 of BC makes the interesting observtion that one can integrate over curves Γ in a Riemann surface $\mathcal{R}$. For example, let $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1$ be the two sheeted Riemann surface for z^2 and consider the curve $\alpha : [0, 4\pi] \to \mathcal{R}$ by

$$\alpha(t) = \begin{cases} re^{it} \in \mathcal{R}_0, & 0 \leq t < 2\pi \\ re^{it} \in \mathcal{R}_1, & 2\pi \leq t < 4\pi. \end{cases}$$

Then

$$\int_\Gamma \sqrt{w}$$

makes sense where $\sqrt{w}$ is the global complex square root function defined on $\mathcal{R}$.

$$\begin{aligned} \int_\Gamma \sqrt{w} &= \int_0^{2\pi} \sqrt{r} e^{it/2}\ ire^{it}\, dt + \int_{2\pi}^{4\pi} \sqrt{r} e^{it/2}\ ire^{it}\, dt \\ &= ir^{3/2} \int_0^{4\pi} e^{3it/2}\, dt \\ &= \frac{2}{3} r^{3/2}\ e^{3it/2} \Big|_0^{4\pi} \\ &= \frac{2}{3} r^{3/2}\ \left[e^{6\pi i} - 1\right] \\ &= 0. \end{aligned}$$

Exercise 4.6 Compute

$$\int_\Gamma \sqrt{w}$$

where $\gamma : [0, \pi] \to \mathcal{R}$ by $\gamma(t) = 3e^{it} \in \mathcal{R}_0$.