Isolated Singularities

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Hopefully you know by now what it means for a function $f : U \to \mathbb{C}$ defined on an open set $U \subset \mathbb{C}$ to be **complex differentiable at a point** $z \in U$ and also to be **differentiable in** U. Here we are going to focus on a function or functions $f : U \to \mathbb{C}$ complex differentiable at every point in the open set U.[1] We consider further, the following special case:

Definition 1 (set definition of isolated singularity) Given a complex differentiable function $f : U \to \mathbb{C}$, we say the point $z_0 \in \mathbb{C}$ is an **isolated singularity** of f if for some $r > 0$ there holds

$$B_r(z_0)\backslash\{z_0\} = \{z \in \mathbb{C} : 0 < |z - z_0| < r\} \subset U.$$

Notice that according to this definition, it may be the case that $z_0 \in U$ (and f is complex differentiable at z_0) or it may be the case that $z_0 \notin U$. This latter case is the case of primary interest.

In the situation when $z_0 \in U$, we say z_0 is a removable (isolated) singularity. For example, we can take a complex differentiable function $g : V \to \mathbb{C}$ and simply consider the restriction $f : U \to \mathbb{C}$ where

$$U = V\backslash\{z_0\} \qquad \text{for some } z_0 \in V \text{ and} \qquad f = g\big|_U.$$

In this case, z_0 becomes an isolated singularity for f, but there is an extension $g : U \cup \{z_0\} \to \mathbb{C}$ for which g is complex differentiable on $V = U \cup \{z_0\}$. Technically, the situation we have just described does not include all removable singularities, but we can use this simple observation as the basis for a definition:

[1]Other terms associated with functions like this are holomorphic, analytic, and conformal (map or function). Each of these words has slightly different connotations associated with it. Some are strictly speaking equivalent to, i.e., just mean the same thing as, "complex differentiabe on the open set U," and others are not quite equivalent.

Definition 2 (removable singularity) Given a complex differentiable function $f : U \to \mathbb{C}$ with an isolated singularity at $z_0 \in \mathbb{C}$, we say z_0 is **removable** or is a **removable singularity** if there exists a complex differentiable function $g : U \cup \{z_0\} \to \mathbb{C}$ with restriction

$$g\big|_U = f.$$

Here is a surprising theorem:

Theorem 1 (removable singularities) If $f : U \to \mathbb{C}$ is complex differentiable and has an isolated singularity at $z_0 \in \mathbb{C}$ for which there exists some $r > 0$ with $B_r(z_0)\backslash\{z_0\} \subset U$ **and** there exists some $M > 0$ for which

$$|f(z)| \le M \qquad \text{for all } z \in B_r(z_0)\backslash\{z_0\}. \tag{1}$$

then f has a removable singularity at z_0.

This is called **Riemann's theorem**. Let me see if I can prove it. For $\epsilon < r$, we have

$$\overline{B_\epsilon(z_0)}\backslash\{z_0\} \subset U.$$

In particular, we can calculate the complex integral

$$\int_{\zeta\in\partial B_\epsilon(z_0)} \frac{f(\zeta)}{\zeta - z} \tag{2}$$

for every $z \in B_\epsilon(z_0)$ including z_0. Remember that by the Cauchy integral formula[2] the value of a complex differentiable function is given by a similar quantity:

$$g(z) = \frac{1}{2\pi i}\int_{\zeta\in\partial B_\delta(w)} \frac{g(\zeta)}{\zeta - z}$$

whenever g is complex differentiable function on an open set containing $\overline{B_\delta(w)}$ and $z \in B_\delta(w)$. I am going to try to use a kind of converse of the Cauchy integral formula.

[2]You might remember also that the Cauchy integral formula followed essentially, from the Cauchy integral theorem by introducing and using some "connecting paths" between $\partial B_\delta(w)$ and an even smaller circle centered at z. This was presented by Ethan Phan, though I think there are still some details that could use some more attention. And finally, the Cauchy integral theorem can be derived from Goursat's lemma, though I don't think we have quite nailed down the details of that derivation—at least as a presentation in a class meeting. I have posted some notes that I think have most of the details.

My Basic claim is that the quantity

$$h(z) = \int_{\zeta \in \partial B_\epsilon(z_0)} \frac{f(\zeta)}{\zeta - z}$$

defines a complex differentiable function on $B_\epsilon(z_0)$. In fact,

$$h(w) - h(z) = \int_{\zeta \in \partial B_\epsilon(z_0)} \left[\frac{f(\zeta)}{\zeta - w} - [\frac{f(\zeta)}{\zeta - z} \right] = \int_{\zeta \in \partial B_\epsilon(z_0)} f(\zeta) \frac{w - z}{(\zeta - w)(\zeta - z)}.$$

Therefore,

$$\frac{h(w) - h(z)}{w - z} = \int_{\zeta \in \partial B_\epsilon(z_0)} \frac{f(\zeta)}{(\zeta - w)(\zeta - z)}.$$

Notice that as $w \to z$, the integrand here converges, uniformly in $\zeta \in \partial B_\epsilon(z_0)$ for z fixed, as $w \to z$ to

$$\frac{f(\zeta)}{(\zeta - z)^2}.$$

Note, in particular, that this "new" integrand is very much nonsingular as a function of $\zeta \in \partial B_\epsilon(z_0)$ for $z \in B_\epsilon(z_0)$ fixed.[3] Thus, it should be easy to believe that

$$\lim_{w \to z} \frac{h(w) - h(z)}{w - z} = \int_{\zeta \in \partial B_\epsilon(z_0)} \frac{f(\zeta)}{(\zeta - z)^2}.$$

The fact that this limit exists happily as some complex number in $\mathbb{C}$ tells us that h is complex differentiable as claimed.

The next question is:

What are the values of this differentiable function h?

Can you guess? If you guessed that for $z \in B_\epsilon(z_0) \backslash \{z_0\}$ there holds $h(z) = 2\pi i f(z)$, then you would be correct. I'm going to try to show this very carefully and with the details.

Let $z \in B_\epsilon(z_0) \backslash \{z_0\}$. This means $z - z_0 \neq 0$. (You can't argue with that!) Let $z - z_0 = |z - z_0| e^{i\theta}$ so that

$$u = \frac{z - z_0}{|z - z_0|} = e^{i\theta}$$

has unit length and look at Figure 1.

[3]This kind of integrand, is not entirely "new" becuase you see this kind of thing in the formula for the derivatives given in the "generalized" Cauchy integral formula, or that is to say, the Cauchy integral formula(s) for the derivatives.

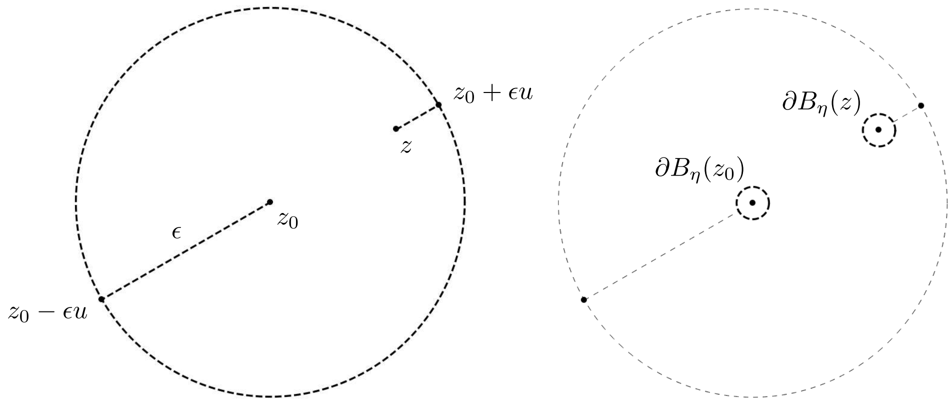

Figure 1: A disk $B_\epsilon(z_0) \subset U$ and a point $z \in B_\epsilon(z_0)\backslash\{z_0\}$. A diameter is determined by $z - z_0 = |z - z_0|u$ (left) and two disjoint compactly constained subdiscs $B_\eta(z_0)$ and $B_\eta(z)$ are determined by $\eta = \min\{|z - z_0|, (\epsilon - |z - z_0|)\}/3$ (right).

Let η satisfy

$$0 < \eta < \min\left\{\frac{|z - z_0|}{3}, \frac{\epsilon - |z - z_0|}{3}\right\}. \tag{3}$$

Then

$$\overline{B_\eta(z_0)}, \overline{B_\eta(z)} \subset B_\epsilon(z_0) \qquad \text{and} \qquad \overline{B_\eta(z_0)} \cap \overline{B_\eta(z)} = \phi.$$

Finally, we modify the closed contour given by the following concatenation:

$$\begin{aligned}
\beta_1(t) &= z_0 + \epsilon\ e^{i(\theta+t)}, \qquad 0 \le t \le \pi \\
\beta_2(t) &= (1-t)(z_0 - \epsilon u) + t(z_0 - \eta u), \qquad 0 \le t \le 1 \\
\beta_3(t) &= z_0 + \eta\ e^{i(\theta+\pi-t)}, \qquad 0 \le t \le 2\pi \\
\beta_4(t) &= (1-t)(z_0 - \eta u) + t(z_0 - \epsilon u), \qquad 0 \le t \le 1 \\
\beta_5(t) &= z_0 + \epsilon\ e^{it}, \qquad \theta + \pi \le t \le \theta + 2\pi \\
\beta_6(t) &= (1-t)(z_0 + \epsilon u) + t(z + \eta u), \qquad 0 \le t \le 1 \\
\beta_7(t) &= z + \eta\ e^{i(\theta-t)}, \qquad 0 \le t \le 2\pi \\
\beta_8(t) &= (1-t)(z + \eta u) + t(z_0 + \epsilon u), \qquad 0 \le t \le 1.
\end{aligned}$$

Exercise 1 Simplify the formulas for β_j, $j = 1, 2, \ldots, 8$ and verify that the concatenation above is a closed curve.

Our modification is made for each τ with $0 < \tau < \pi/2$, and we are interested ultimately in the limit as $\tau \searrow 0$. To be precise, we consider the concatenation

$$
\begin{aligned}
\gamma_1(t) &= z_0 + \epsilon\ e^{i(\theta+t)}, \qquad \tau \le t \le \pi - \tau \\
\gamma_2(t) &= (1-t)(z_0 + \epsilon e^{i(\theta+\pi-\tau)}) + t(z_0 + \eta e^{i(\theta+\pi-\tau)}), \qquad 0 \le t \le 1 \\
\gamma_3(t) &= z_0 + \eta\ e^{i(\theta+\pi-t)}, \qquad \tau \le t \le 2\pi - \tau \\
\gamma_4(t) &= (1-t)(z_0 + \eta e^{i(\theta-\pi+\tau)}) + t(z_0 + \epsilon e^{i(\theta-\pi+\tau)}), \qquad 0 \le t \le 1 \\
\gamma_5(t) &= z_0 + \epsilon\ e^{it}, \qquad \theta + \pi + \tau \le t \le \theta + 2\pi - \tau \\
\gamma_6(t) &= (1-t)(z_0 + \epsilon e^{i(\theta-\tau)}) + t(z + \eta e^{i(\theta-\psi)}), \qquad 0 \le t \le 1 \\
\gamma_7(t) &= z + \eta\ e^{i(\theta-t)}, \qquad \psi \le t \le 2\pi - \psi \\
\gamma_8(t) &= (1-t)(z + \eta e^{i(\theta+\psi)}) + t(z_0 + \epsilon e^{i(\theta+\tau)}), \qquad 0 \le t \le 1
\end{aligned}
$$

where

$$\psi = \arccos\left(\frac{\epsilon\cos\tau - |z - z_0|}{\sqrt{\epsilon^2\cos^2\tau - 2\epsilon|z - z_0|\cos\tau + |z - z_0|^2}}\right)$$

and $\arccos : [-1, 1] \to [0, \pi]$ is the standard real inverse cosine. See figure 2.

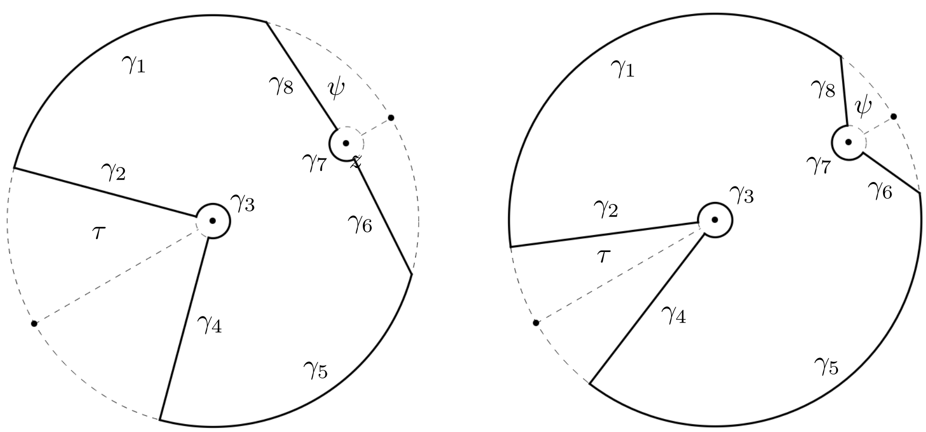

Figure 2: Simple closed curves bounded by arcs of circles and straight line segments. A larger value $\tau = \pi/4$ is illustrated on the left where the intersection angle ψ is a little larger than $\pi/2$. A smaller value $\tau = \pi/8$ is illustratted on the right.

Exercise 2 Show the concatenation illustrated in Figure 2 is a simple closed curve.

Exercise 3 Draw the limiting contour for $\tau = \pi/2$.

The simple closed curve given by this modification is the boundary of a bounded open set J (given by the Jordan curve theorem, but also easily defined precisely in this case). The closure $\overline{J}$ of the region J furthermore is contained in an open set $V = B_r(z_0)\backslash\{z_0, z\}$ on which the function $\phi : V \to \mathbb{C}$ by

$$\phi(\zeta) = \frac{f(\zeta)}{\zeta - z}$$

is complex differentiable. By the Cauchy integral theorem we have

$$\sum_{j=1}^{8} \int_{\gamma_j} \phi = 0 \tag{4}$$

for all τ with $0 < \tau < \pi/2$.

Next we compute some limits.

$$\int_{\gamma_1} \phi = \int_{\theta+\tau}^{\theta+\pi-\tau} \frac{f(z_0 + \epsilon e^{it})}{z_0 - z + \epsilon e^{it}} \, i\epsilon e^{it} \, dt.$$

Notice the integrand does not depend on τ. Therefore,

$$\lim_{\tau \to 0} \int_{\gamma_1} \phi = \int_{\theta}^{\theta+\pi} \frac{f(z_0 + \epsilon e^{it})}{z_0 - z + \epsilon e^{it}} \, i\epsilon e^{it} \, dt.$$

Similarly,

$$\lim_{\tau \to 0} \int_{\gamma_5} \phi = \int_{\theta+\pi}^{\theta+2\pi} \frac{f(z_0 + \epsilon e^{it})}{z_0 - z + \epsilon e^{it}} \, i\epsilon e^{it} \, dt.$$

Putting these together,

$$\lim_{\tau \searrow 0} \left(\int_{\gamma_1} \phi + \int_{\gamma_5} \phi \right) = h(z).$$

This is a value of interest for us. Moving to the next portion of the contour indicated in Figure 2,

$$\begin{aligned}
\int_{\gamma_2} \phi &= \int_0^1 \frac{f(z_0 + [(1-t)\epsilon + t\eta] e^{i(\theta+\pi-\tau)})}{z_0 - z + [(1-t)\epsilon + t\eta] e^{i(\theta+\pi-\tau)}} \, [-(\epsilon - \eta)] e^{i(\theta+\pi-\tau)} \, dt \\
&= (\epsilon - \eta) \int_0^1 \frac{f(z_0 - [(1-t)\epsilon + t\eta] e^{i(\theta-\tau)})}{z_0 - z - [(1-t)\epsilon + t\eta] e^{i(\theta-\tau)}} \, e^{i(\theta-\tau)} \, dt.
\end{aligned}$$

Here the dependence on τ is entirely inside the integrand and (1) the argument of f is bounded away from the singularity at z_0 and (2) the denominator is also uniformly bounded away from zero. Thus, the integrand converges uniformly for $0 \leq t \leq 1$ as $\tau \to 0$ to the nonsingular value

$$\frac{f(z_0 - [(1-t)\epsilon + t\eta]e^{i\theta})}{z_0 - z - [(1-t)\epsilon + t\eta]e^{i\theta})}\, e^{i\theta},$$

and

$$\lim_{\tau \searrow 0} \int_{\gamma_2} \phi = \int_{\beta_2} \phi.$$

Also,

$$\int_{\gamma_4} \phi = \int_0^1 (\eta - \epsilon) \frac{f(z_0 - [(1-t)\eta + t\epsilon]e^{i(\theta+\tau)})}{z_0 - z - [(1-t)\eta + t\epsilon]e^{i(\theta+\tau)})}\, e^{i(\theta+\tau)}\, dt,$$

and

$$\lim_{\tau \searrow 0} \int_{\gamma_4} \phi = \int_{\beta_4} \phi = -\int_{\beta_2} \phi.$$

We conclude,

$$\lim_{\tau \searrow 0} \left(\int_{\gamma_2} \phi + \int_{\gamma_4} \phi \right) = 0.$$

Using the same approach presented above, it is easy to see

$$\lim_{\tau \searrow 0} \int_{\gamma_3} \phi = \int_{\beta_3} \phi = -\int_{\zeta \in \partial B_\eta(z_0)} \frac{f(\zeta)}{\zeta - z}.$$

Also, we can compute the limit as $\tau \searrow 0$ of the intersection angle ψ:

$$\lim_{\tau \searrow 0} \arccos \left(\frac{\epsilon \cos \tau - |z - z_0|}{\sqrt{\epsilon^2 \cos^2 \tau - 2\epsilon |z - z_0| \cos \tau + |z - z_0|^2}} \right) = \arccos(1) = 0.$$

From this it follows that

$$\lim_{\tau \searrow 0} \int_{\gamma_6} \phi = \int_{\beta_6} \phi$$

and

$$\lim_{\tau \searrow 0} \int_{\gamma_8} \phi = \int_{\beta_8} \phi = -\int_{\beta_6} \phi.$$

In particular,

$$\lim_{\tau \searrow 0} \left(\int_{\gamma_6} \phi + \int_{\gamma_8} \phi \right) = 0.$$

We have one more integral to consider, namely lucky

$$\int_{\gamma_7} \phi.$$

It should come as no surprise that

$$\lim_{\tau \searrow 0} \phi = -\int_{\zeta \in \partial B_\eta(z)} \frac{f(\zeta)}{\zeta - z} = -2\pi i f(z)$$

by the Cauchy integral formula. Overall, we conclude

$$h(z) = \int_{\zeta \in \partial B_\eta(z_0)} \frac{f(\zeta)}{\zeta - z} + 2\pi i f(z).$$

and this identity holds for all η satisfying (3) and, in particular, for all positive η small enough. This means we can take the limit as η tends to zero, and since the denominator $\zeta - z$ remains bounded away from zero for $\zeta \in \partial B_\eta(z_0)$, it follows that the integrand $\phi(\zeta)$ is bounded by some positive number A independent of η in the limit. That is, we have an estimate

$$\left| \int_{\zeta \in \partial B_\eta(z_0)} \frac{f(\zeta)}{\zeta - z} \right| \le A(2\pi\eta) \to 0 \qquad \text{as} \qquad \eta \searrow 0.$$

We conclude $h : B_\epsilon(z_0) \to \mathbb{C}$ is a complex differentiable function with $h(z) = 2\pi i f(z)$ for $z \neq z_0$, or in other words $g : U \cup \{z_0\} \to \mathbb{C}$ by

$$g(z) = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\partial B_\epsilon(z_0)} \frac{f(\zeta)}{\zeta - z}, & z \in B_\epsilon(z_0) \\ f(z), & z \in U \end{cases}$$

is a well-defined complex differentiable function with

$$g\big|_U \equiv f.$$

The function f has a removable singularity at $z = z_0$. $\square$

Exercise 4 Give an example of a function $\phi : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2$ which is (infinitely) differentiable, i.e., all partial derivatives of both coordinate functions exist at all points in the punctured plane, and is bounded but cannot be extended continuously to the entire plane (much less differentiably).

Riemann's theorem suggests the following hierarchy of isolated singularities:

Definition 3 (structural definition of isolated singularities; poles) If $f : U \to \mathbb{C}$ is complex differentiable and has an isolated singularity at $z_0 \in \mathbb{C}$ and $m \in \mathbb{N} = \{1, 2, 3, \ldots\}$, then we say f has a **singularity of order** m at z_0 if for some $r > 0$

$$\sup_{z \in B_r(z_0)} |(z - z_0)^{m-1} f(z)| = \infty \qquad \text{but} \qquad \sup_{z \in B_r(z_0)} |(z - z_0)^m f(z)| < \infty.$$

In this case, f is also said to have a **pole** or order m at z_0.

Note that in the definition of isolated singularity of order m when we take these suprema[4] it is implicit that only values of z in the domain U of f are under consideration. Thus, we mean the supremum is effectively taken over $B_r(z_0) \backslash \{z_0\}$, though we didn't write this. It is also assumed here that $B_r(z_0) \backslash \{z_0\} \subset U$, though we didn't say this. The same (implicit) conventions/assumptions will continue to be used below.

We have talked about two kinds of isolated singularities so far: removable singularities and poles. It turns out that these are not all the isolated singularities.

Definition 4 (essential singularity) Given $f : U \to \mathbb{C}$ a complex differentiable function defined on an open set U in $\mathbb{C}$, if $z_0 \in \mathbb{C}$ is an isolated singularity of f but z_0 is not removable or a pole, then z_0 is said to be an **essential singularity** of f.

Exercise 5 Show $f : \mathbb{C} \backslash \{0\} \to \mathbb{C}$ by

$$f(z) = e^{1/z}$$

has an essential singularity at $z_0 = 0$.

If f has a singularity of order $m \in \mathbb{N}$ at z_0, then for each $j \in \mathbb{N}$

$$\sup_{z \in B_r(z_0)} |(z - z_0)^{m-1-j} f(z)| = \infty \qquad \text{but} \qquad \sup_{z \in B_r(z_0)} |(z - z_0)^{m+j} f(z)| < \infty.$$

[4]A **supremum** is a least upper bound if there exists a (finite) least upper bound and is $+\infty$ otherwise. The difference between a supremum and a least upper bound is that a supremum can take the value $+\infty$ and thus takes its values among the extended real numbers $\mathbb{R} \cup \{+\infty\}$ while a least upper bound is always a real number.

Also, the functions $Q_m : B_r(z_0)\backslash\{z_0\} \to \mathbb{C}$ (for every $r > 0$ and $z_0 \in \mathbb{C}$) given by

$$Q_m(z) = \frac{1}{(z - z_0)^m}$$

are examples of functions with an isolated singularity of order m, i.e., a pole. More generally, if $m, n \in \mathbb{N}$ and $a_{-m}, a_{-(m-1)}, \ldots, a_{-1}, a_0, a_1, \ldots, a_n \in \mathbb{C}$ and $a_{-m} \neq 0$, then $f : \mathbb{C}\backslash\{z_0\} \to \mathbb{C}$ (for every $z_0 \in \mathbb{C}$) by

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-(m-1)}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots + a_n(z - z_0)^n = \sum_{j=-m}^{n} a_j(z - z_0)^j$$

are examples of functions with an isolated singularity of order m at z_0.

Exercise 6 (more examples of meromorphic functions) If $z_1, z_2, \ldots, z_k$ are distinct complex numbers, $a_1, a_2, \ldots, a_k \in \mathbb{C}\backslash\{0\}$, and $m_1, m_2, \ldots, m_k \in \mathbb{N}$, then $f : \mathbb{C}\backslash\{z_1, z_2, \ldots, z_k\} \to \mathbb{C}$ by

$$f(z) = \sum_{j=1}^{k} \frac{a_j}{(z - z_j)^{m_j}}$$

has isolated singularities precisely at the points $z_1, z_2, \ldots, z_j$. What is the positive radius r_j, say the largest such radius, on which

$$f\Big|_{B_{r_j}(z_j)} \qquad \text{is complex differentiable?}$$

Roughly speaking, a meremorphic function is a complex differentiable function with only poles as isolated singularities. There are some technicalities involved, but here is a possible/simple definition:

Definition 5 (meromorphic) Given $f : U \to \mathbb{C}$ a complex differentiable function on on open set $U \subset \mathbb{C}$, we say f is **meromorphic** if each z_0 in the *interior of the closure* of U is either a removable (isolated) singularity or a pole of (some) order m of the restriction

$$f\Big|_{U\backslash\{z_0\}}.$$

In other words, for each $z_0 \in \text{int}(\overline{U})$, the point z_0 is an isolated singularity of f and that isolated singularity is not an essential singularity.

Exercise 7 (moromerphic functions) Just for fun, let's say a function $f : U \to \mathbb{C}$, complex differentiable on an open set $U \subset \mathbb{C}$, is **moremerphic** if each isolated singularity $z_0 \in \overline{U}$ is either a removable singularity or a pole. What is the difference between a moremerphic function and a meromorphic function?

Riemann's theorem gives the following:

Exercise 8 (basic structure theorem for meromorphic functions) If $f : B_r(z_0)\backslash\{z_0\} \to \mathbb{C}$ is complex differentiable and has a pole of order m at z_0, then there exists a complex differentiable function $g : B_r(z_0) \to \mathbb{C}$ with $g(z_0) \neq 0$ and

$$f = Q_m \; g\big|_{B_r(z_0)\backslash\{z_0\}}$$

where as above

$$Q_m(z) = \frac{1}{(z - z_0)^m}.$$

Hint: Show that if $g(z_0) = 0$, then there exists a complex differentiable function g_1 for which $g = (z - z_0)g_1$.

Now if you believe[5] every complex differentiable function $g : U \to \mathbb{C}$ is locally represented by a convergent power series, i.e., if U is open and $z_0 \in U$, then there is some $\epsilon > 0$ for which $B_\epsilon(z_0) \subset U$ and for $z \in B_\epsilon(z_0)$ there holds

$$g(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

where the sequence of coefficients $a_1, a_2, a_3, \ldots$ consists of complex numbers and the series on the right converges in the sense that

$$\lim_{k\to\infty} \sum_{j=0}^{k} a_j (z - z_0)^j = g(z),$$

then the structure thorem gives this: Whenever z_0 is a pole of order m as in Exercise 8, then f has a local representation

$$f(z) = \frac{1}{(z - z_0)^m} \sum_{j=0}^{\infty} a_j (z - z_0)^j = \sum_{j=-m}^{\infty} b_j (z - z_0)^j$$

[5]We have not yet shown this.

where $b_j = a_{j+m}$. That is,

$$f(z) = \frac{b_{-m}}{(z-z_0)^m} + \frac{b_{-(m-1)}}{(z-z_0)^{m-1}} + \cdots + \frac{b_{-1}}{z-z_0} + \sum_{j=0}^{\infty} b_j (z-z_0)^j. \tag{5}$$

Furthermore, if you believe the coefficients in the analytic/series representation are unique with for example

$$a_j = \frac{g^{(j)}(z_0)}{j!} \qquad \text{for } j = 0, 1, 2, 3, \ldots, \tag{6}$$

then the construction above defines a unique sequence of numbers and you can identify one of them as special:

Definition 6 (residue) Given $f : U \to \mathbb{C}$ complex differentiable on an open set $U \subset \mathbb{C}$ with a pole at $z_0 \in \mathbb{C}$, the **residue** of f at z_0 is defined by

$$\text{res}_{z_0}(f) = b_{-1} \qquad \text{where } b_{-1} \text{ is given in (5).}$$

In fact,

$$\text{res}_{z_0}(f) = \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[\frac{1}{(m-1)!} \, (z-z_0)^m f(z) \right].$$

You are encouraged to try out my definition and formula for the residue on the examples of meromorphic functions given above and others you can find yourself. They might even be correct. If so, then the following may be of interest:

Theorem 2 (the residue theorem) If

1. Γ is a simple closed contour, meaning continuous complex valued functions can be integrated along this curve,

2. The bounded component V of the complement $\mathbb{C} \backslash \Gamma$ of Γ is compactly contained in an open set $U \subset \mathbb{C}$, i.e., $\overline{V} \subset U$, and

3. $f : U \backslash \{z_1, z_2, \ldots, z_k\}$ is complex differentiable and meromorphic with poles precisely at $z_1, z_2, \ldots, z_k \in V$,

then

$$\int_\Gamma f = 2\pi i \sum_{j=1}^{k} \text{res}_{z_j}(f).$$