Special Project: Mike's Hyperbola(s)
Due Wednesday, May 3, 2023

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Michael Ani presented a solution of Problem 5 from Assignment 2. The statement of the problem was as follows:
Problem: (Exercises 1.5.5,6 and 1.6.14 in BC) For each part draw the points z satisfying the relation in the complex plane:

(a) $|z - 1 + i| = 1$.

(b) $|z + i| \leq 3$.

(c) $|z - 4i| \geq 4$.

(d) $|z - 1| = |z + i|$.

(e) $z^2 + \overline{z}^2 = 2$.

The last relation $\{z \in \mathbb{C} : z^2 + \overline{z}^2 = 2\}$ turns out to be a hyperbola with Euclidean equation

$$x^2 - y^2 = 1.$$

Thus, the complex equation of the Euclidean hyperbola $x^2 - y^2 = 1$ can be taken to be

$$z^2 + \overline{z}^2 = 2. \tag{1}$$

I pointed out that while most of the other relations in Problem 5, especially the ones involving circles, can be immediately recognized in terms of familiar Euclidean characterizations of sets, e.g., the set of all points at a fixed distance from a given point, the complex relation for a hyperbola did not lend itself to such an immediate interpretation (as the set of all points the difference of whose distances from two fixed points, in this case $\pm\sqrt{2}+0i$, is fixed, in this case at $2a = 2$—at least as far as I could see. In particular, the geometric role played by the position of the focal distance $\sqrt{2}$, though the numerical value 2 appears prominently in (1), does not strike me as immediately clear.

Leo Wang ponted out an alternative form of (1) is

$$\mathrm{Re}(z^2) = 1, \tag{2}$$

and I could then see from this an immediate geometric interpretation in terms of the complex square function. I leave that as part of this project.

There are a couple questions that, it seems to me, one should ask immediately:

1. How does one "see" the Euclidean characterization of the hyperbola $x^2 - y^2 = 1$, as the collection of points the difference of whose distances to $\pm\sqrt{2}$ is 2, in the corresponding complex relation (1).

2. What precisely does one need to know about the complex square function in order to see the complex hyperbola as represented in the relation (2)?

Beyond these questions, one might ask for the complex relation defining a more general hyperbola in standard postion

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or even more general hyperbolas—with the opposite orientation $y^2/b^2 - x^2/a^2 = 1$, an additional affine translation, or even with a rotation. I think I may have made some comments, at least of minor relevance, in my notes.