Complex Analysis: Final

Harvard University — Fall 1997 — Math 113

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1. Let $f_n \to f$ uniformly on compact subsets of an open connected set $\Omega \subset \mathbb{C}$, where f_n is analytic, and f is not identically equal to zero.

   (a) Show if $f(w) = 0$ then we can write $w = \lim z_n$, where $f_n(z_n) = 0$ for all n sufficiently large.

   Answer: it suffices to prove that for each $r > 0$, f_n has a zero in $B(w, r)$ for all $n \gg 0$. But $f|B(w, r)$ is not identically zero, so this follows by Hurwitz's theorem (Ahlfors p. 178).

   (b) Does this result hold if we only assume Ω is open?

   Answer: no; for example, take $\Omega = \mathbb{C} - \mathbb{R}$, and let $f_n = 1$ in the lower half-plane and $1/n$ in the upper half-plane.

2. Let $f(z) = (az+b)/(cz+d)$ be a Möbius transformation. Show the number of rational maps $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

$$g(g(g(g(g(z))))) = f(z)$$

   is 1, 5 or ∞. Explain how to determine which alternative holds for a given f.

   Answer: Since $\deg(f) = 1 = \deg(g)^5$, g is also a Möbius transformation. After conjugating both sides of the equation, we can assume either:

   f is the identity $f(z) = z$; in which case g has infinitely many solutions, e.g. $g(z) = \exp(2\pi i/5)z + a$, for any a;
   f is hyperbolic, $f(z) = \lambda z$, with λ different from 0 and 1; then $g(z) = \lambda^{1/5} z$ gives 5 solutions; or
   f is parabolic, $f(z) = z + 1$; in which case $g(z) = z + 1/5$ is the unique solution.

   Normalizing so $ad - bc = 1$, and assuming $f(z)$ is not the identity, the number of solutions is 1 if $(a + d)^2 = 4$, and 5 otherwise.

3. Find all meromorphic functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$ such that $f(1) = 1$ and

$$f(z + 1) = zf(z).$$

   Answer: $f(z) = \Gamma(z)h(e^{2\pi i z})$ where h is any arbitrary meromorphic function on $\mathbb{C}^*$ with $h(1) = 1$. To see this, note that $f(z)/\Gamma(z) = g(z)$ satisfies $g(z + 1) = g(z)$, so $h(z) = g(\log z/(2\pi i))$ is a well-defined meromorphic function for $z \neq 0$.

4. Let $\sum a_n z^n$ be the Taylor series for $\tan(z)$ at $z = 0$.

   (a) What is the radius of convergence of this power series?

   Answer: $R = \pi/2$.

   (b) Give an explicit value of N such that $\tan(1)$ and $\sum_0^N a_n$ agree to 1000 decimal places. Justify your answer.

   Answer: $N = 5700$, for example, will work. Let $r = 1.5$ and observe that on the circle $S(0, r)$ we have, setting $w = e^{iz}$,

$$|\tan(z)| = \frac{|w||w - 1/w|}{|w^2 + 1|} \le \frac{2e^3}{\cos(1.5)} \le 600.$$

   (Here $|w^2 + 1| > \cos(1.5)$ because $|z| < 1.5 \implies |\arg(w^2)| \le 1.5 \implies |w^2+1| \ge \cos(1.5)$.) By Cauchy's estimate, $|a_n| \le 600r^{-n} \le 600(1.5)^{-n}$, so

$$\sum_N^\infty |a_n| < 600r^{-N}/0.5 < 10^{-1000}$$

   if we take $N > (\log(1200) + 1000 \log(10))/\log(1.5) = 5696.3\ldots$.

5. Evaluate:

$$\int_{-\infty}^{\infty} \frac{x^6}{(1 + x^4)^2}\, dx.$$

   Answer: $I = 3\pi\sqrt{2}/8$. The integrand has poles at the roots of $x^4 = 1$; the two roots $x = (i \pm 1)/\sqrt{2}$ lie in the upper half-plane. The residues at these two poles are $3(i \pm 1)\sqrt{2}/32$, and the integral is $2\pi i$ times the sum of the residues in $\mathbb{H}$.

6. Let $f : \mathbb{C} \to \mathbb{C}$ be analytic and let $U \subset \mathbb{C}$ be a bounded region. Suppose $|f(z)|$ is constant on ∂U. Show that either f is constant, or f has a zero in U.

   Answer: Assume f has no zero in U, and let M be the constant value assumed by $|f|$ on ∂U; then $|f| \le M$ throughout U, by the maximum principle, so $M > 0$. Similarly $1/f(z)$ is analytic in U, so $|1/f(z)| \le 1/M$ in U and thus $|f(z)|$ is constant. By the open mapping theorem, f must be constant.

7. Compute the Laurent series centered at $z = 0$ such that

$$\sum_{-\infty}^{\infty} a_n z^n = \frac{1}{z(z-1)(z-2)}$$

   in the region $1 < |z| < 2$.

   Answer: using partial fractions we find

$$f(z) = \frac{1}{z(z-1)(z-2)} = \frac{1}{2z} + \frac{1}{1-z} + \frac{1}{2(z-2)};$$

   expanding each term in a Laurent series valid in the given region, we find

$$f(z) = \cdots - \frac{1}{z^4} - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{2z} - \frac{1}{4} - \frac{z}{8} - \frac{z^2}{16} - \frac{z^3}{32} - \cdots$$

8. Show for any polynomial $p(z)$ there is a z with $|z| = 1$ such that $|p(z) - 1/z| \ge 1$.

   Answer: $2\pi i = \int_{S^1} (p(z) - 1/z)\, dz$, and since the length of S^1 is 2π, the average value of the absolute value of the integrand must be at least one.

9. Let $U = \{z : 0 < |z| < 1 \text{ and } 0 < \arg(z) < \alpha\}$, where $0 < \alpha < 2\pi$. Find a formula for a conformal homeomorphism $f : U \to \mathbb{H}$, where $\mathbb{H} = \{z : \mathrm{Im}(z) > 0\}$ is the upper half-plane.

   Answer: $f(z) = -z^{\pi/\alpha} - z^{-\pi/\alpha}$. The map $z \mapsto z^{\pi/\alpha}$ sends U to $\Delta \cap \mathbb{H}$, and the map $z \mapsto -z - 1/z$ sends $\Delta \cap \mathbb{H}$ to $\mathbb{H}$.

10. Express

$$f(z) = \sum_{-\infty}^{\infty} \frac{1}{(z-n)^4}$$

   in closed form, using trigonometric functions.

   Answer: We know from Ahlfors that

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{-\infty}^{\infty} \frac{1}{(z-n)^2};$$

   differentiating both sides twice, we find

$$f(z) = \sum_{-\infty}^{\infty} \frac{1}{(z-n)^4} = \frac{\pi^4}{\sin^4(\pi z)} - \frac{2\pi^4}{3\sin^2(\pi z)}$$

   (This formula also shows $\sum_1^\infty 1/n^4 = \pi^4/90$, by comparing the constant terms in the Laurent expansions on both sides.)