Cauchy's Theorem and Goursat's Lemma

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Let us begin with a definition:

Definition 1 (Cauchy domain) A bounded open subset $\Omega \subset \mathbb{C}$ is a **Cauchy domain** if $\Gamma = \partial\Omega$ is a piecewise C^1 simple closed curve.

In order to understand this definition naturally requires that one know what **piecewise** C^1 means. Also, implicit in the definition is the use of the **Jordan curve theorem**. I will not discuss those topics here. The basic result under discussion here is the following:

Theorem 1 (Cauchy's theorem) If U is an open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ is a complex differentiable function, then

$$\int_\Gamma f = 0 \qquad \text{for every Cauchy domain } \Omega \subset U \text{ with } \Gamma = \partial\Omega \subset U. \tag{1}$$

1 Proof when f' is continuous

Recall Green's theorem from multivariable calculus:

Theorem 2 (Green's theorem) If

(i) Ω is a bounded open subset of $\mathbb{R}^2$,

(ii) $\partial\Omega$ a piecewise C^1 simple closed curve,

(iii) U is an open set in $\mathbb{R}^2$, and

(iv) $\mathbf{v} : U \to \mathbb{R}^2$ is a C^1 vector field on U, then

$$\int_{\partial\Omega} \mathbf{v} \cdot T = \int_{\Omega} \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right).$$

where $\mathbf{v} = (v_1, v_2)$.

I will mention that condition **(iv)** $\mathbf{v} \in C^1(U \to \mathbb{R}^2)$ means the first partial derivatives of the component functions $v_1, v_2 : U \to \mathbb{R}$ are continuous, i.e.,

$$\frac{\partial v_1}{\partial x}, \ \frac{\partial v_1}{\partial y}, \ \frac{\partial v_2}{\partial x}, \ \frac{\partial v_2}{\partial y} \in C^0(U).$$

Exercise 1 Give a simple proof of Cauchy's theorem using Green's theorem under the additional assumption that f' is continuous. Note that $f = u + iv$ satisfies f' is continuous on U if and only if the (first) partial derivatives of the real and imaginary parts as functions of the real variables x and y where $z = x + iy$ satisfy

$$\frac{\partial u}{\partial x}, \ \frac{\partial u}{\partial y}, \ \frac{\partial v}{\partial x}, \ \frac{\partial v}{\partial y} \in C^0(U).$$

2 Another "special" case

If f' is continuous, and the other hypotheses of Cauchy's theorem hold, then the conclusion (1) can be obtained using Green's theorem about circulation integrals for real vector fields. Another situation in which the assertion

$$\int_{\Gamma} f = 0$$

of Cauchy's theorem may be obtained is when there is a global complex **antiderivative** $g : U \to \mathbb{C}$, that is a complex differentiable function for which $g' = f$. In this case a version of the fundamental theorem of calculus gives the result: Parameterizing

Γ by $\alpha : [a,b] \to \Gamma$ with α injective on $[a,b)$ and $\alpha(b) = \alpha(a)$, we can write

$$\begin{aligned}
\int_\Gamma f &= \int_a^b f \circ \alpha(t)\, \alpha'(t)\, dt \\
&= \int_a^b g' \circ \alpha(t)\, \alpha'(t)\, dt \\
&= \int_a^b \frac{d}{dt}[g \circ \alpha(t)]\, dt \qquad (2) \\
&= g \circ \alpha\Big|_{t=a}^{b} \qquad (3) \\
&= g(\alpha(b)) - g(\alpha(a)) \\
&= 0.
\end{aligned}$$

Exercise 2 Prove a version of the chain rule for real derivatives of complex valued compositions $g \circ \alpha$ of functions $\alpha : [a,b] \to U$ and $g : U \to \mathbb{C}$ justifying (2).

Exercise 3 Prove a version of the fundamental theorem of calculus for real integrals of continuous complex valued functions $\phi : [a,b] \to \mathbb{C}$ justifying (3).

We will use this special case of Cauchy's theorem below.

3 Gorsat domains

The way I learned it, Goursat's lemma is the following:

Lemma 1 If R is a rectangular domain in $\mathbb{C}$, i.e., there is a (center) point $z_0 \in \mathbb{C}$ and width and height values $a, b > 0$ such that

$$R = R_{a,b}(z_0) = \left\{ z \in \mathbb{C} : |\operatorname{Re}(z) - \operatorname{Re}(z_0)| < \frac{a}{2},\ |\operatorname{Im}(z) - \operatorname{Im}(z_0)| < \frac{b}{2}, \right\},$$

then

$$\int_{\partial R} f = 0 \qquad \text{for every complex differentiable function } f : U \to \mathbb{C}$$

where U is an open subset of $\mathbb{C}$ with $\overline{R} \subset U$.

Note that this is a special case of Cauchy's theorem. I will present a proof for the special case $a = b$ and $z_0 = 0$.

Proof: Let us assume by way of contradiction that

$$\int_{\partial R} f \neq 0.$$

This means there is some $\epsilon > 0$ for which

$$\left| \int_{\partial R} f \right| \geq \epsilon. \tag{4}$$

Notice that if $R = R_{a,a}(0) = \{z \in \mathbb{C} : |\operatorname{Re}(z)|, |\operatorname{Im}(z)| < a/2\}$, then there are four square subdomains

$$\begin{aligned}
R_1 &= R_{a/2,a/2}(a/2 + ia/2) = \{z \in \mathbb{C} : |\operatorname{Re}(z) - a/2|, |\operatorname{Im}(z) - a/2| < a/2\} \\
R_2 &= R_{a/2,a/2}(-a/2 + ia/2) = \{z \in \mathbb{C} : |\operatorname{Re}(z) + a/2|, |\operatorname{Im}(z) - a/2| < a/2\} \\
R_3 &= R_{a/2,a/2}(-a/2 - ia/2) = \{z \in \mathbb{C} : |\operatorname{Re}(z) + a/2|, |\operatorname{Im}(z) + a/2| < a/2\} \\
R_4 &= R_{a/2,a/2}(a/2 - ia/2) = \{z \in \mathbb{C} : |\operatorname{Re}(z) - a/2|, |\operatorname{Im}(z) + a/2| < a/2\}
\end{aligned}$$

for which

$$\int_{\partial R} f = \sum_{j=1}^{4} \int_{\partial R_j} f.$$

If we assume

$$\left| \int_{\partial R_j} f \right| < \frac{\epsilon}{4} \qquad \text{for } j = 1, 2, 3, 4,$$

then

$$\left| \int_{\partial R} f \right| \leq \sum_{j=1}^{4} \left| \int_{\partial R_j} f \right| < \epsilon,$$

which constradicts (4). We conclude one of the subdomains $R_{1,j_1} \in \{R_1, R_2, R_3, R_4\}$ must satisfy

$$\left| \int_{\partial R_{1,j_1}} f \right| \geq \frac{\epsilon}{4}.$$

A similar argument applies to the square domain R_{1,j_1} giving a subdomain R_{2,j_2}, which is also a square subdomain of both R and R_{1,j_1} satisfying

$$\left|\int_{\partial R_{2,j_2}} f\right| \geq \frac{\epsilon}{4^2}.$$

Continuing in this way, we obtain a sequence of nested square domains

$$R_{1,j_1} \supset R_{2,j_2} \supset \cdots \supset R_{n,j_n} \supset \cdots$$

where R_{n,j_n} has sidelength $a/2^n$ and satisfies

$$\left|\int_{\partial R_{n,j_n}} f\right| \geq \frac{\epsilon}{4^n}. \tag{5}$$

Furthermore, the centers $z_{0,n}$ of these square domains form a sequence which converges to a point $z_1 \in \mathbb{C}$ satisfying

$$z_1 \in \cap_{n=1}^{\infty} \overline{R_{n,j_n}}.$$

Note that $z_1 \in U$, and therefore, we can find some $\delta > 0$ for which $\eta : B_\delta(z_1) \to \mathbb{C}$ by

$$\eta(z) = f(z) - f(z_1) - f'(z_1)(z - z_1)$$

is a continuous function satisfying

$$\lim_{z \to z_1} \frac{\eta(z)}{z - z_1} = 0. \tag{6}$$

When n is large enough, the entire square domain R_{n,j_n} and its boundary will lie within $B_\delta(z_1)$. This means that in the integral

$$\int_{\partial R_{n,j_n}} f = \int_{z \in \partial R_{n,j_n}} f(z)$$

we can write $f(z) = f(z_1) + f'(z_1)(z - z_1) + \eta(z)$. Therefore,

$$\int_{\partial R_{n,j_n}} f = f'(z_1) \int_{\partial R_{n,j_n}} 1 + f'(z_1) \int_{z \in \partial R_{n,j_n}} (z - z_1) + \int_{\partial R_{n,j_n}} \eta.$$

Let's take each of the integrals on the right separately. First of all, there is an entire function $g_1(z) = z$ for which $g_1'(z) \equiv 1$. This means

$$\int_{\partial R_{n,j_n}} 1 = 0.$$

Next, there is an entire function $g_2(z) = (z - z_1)^2/2$ for which $g_2'(z) = (z - z_1)$. Therefore,

$$\int_{z \in \partial R_{n,j_n}} (z - z_1) = 0$$

for the same reason. Thus, we only need to consider the third integral for which

$$\int_{\partial R_{n,j_n}} f = \int_{\partial R_{n,j_n}} \eta.$$

We estimate:

$$\left| \int_{\partial R_{n,j_n}} \eta \right| \leq \frac{4a}{2^n} \max_{z \in \partial R_{n,j_n}} |\eta(z)|$$

where $4a/2^n$ is the perimeter of the square $\partial R_{n,j_n}$. On the other hand, for any $\rho > 0$, we can take n large enough so that

$$|z - z_1| \leq \frac{a}{2^{n+1}} \sqrt{2} \qquad \text{for } z \in \partial R_{n,j_n}$$

and (consequently due to (6))

$$|\eta(z)| \leq \rho \; |z - z_1| < \frac{a\sqrt{2}}{2^{n+1}} \; \rho.$$

Therefore, for all such large enough n

$$\max_{z \in \partial R_{n,j_n}} |\eta(z)| \leq \frac{a\sqrt{2}}{2^{n+1}} \; \rho$$

and

$$\left| \int_{\partial R_{n,j_n}} \eta \right| \leq \frac{4a}{2^n} \; \frac{a\sqrt{2}}{2^{n+1}} \; \rho = \frac{2a^2\sqrt{2}}{2^{2n}} \; \rho.$$

Putting this estimate together with (5) we have

$$\frac{\epsilon}{4^n} \leq \left| \int_{\partial R_{n,j_n}} \eta \right| \leq \frac{2a^2\sqrt{2}}{2^{2n}} \; \rho$$

and

$$\epsilon \leq 2a^2\sqrt{2} \; \rho$$

where $\epsilon > 0$ and $a > 0$ are fixed and $\rho > 0$ is arbitrary. This is a contradiction, and this contradiction means our assumption (4) cannot be true. It must be the case that

$$\int_{\partial R} f = 0. \qquad \Box$$

Exercise 4 Prove Goursat's lemma (for a general rectangle).

Essentially the same argument applies to any Goursat domain:

Definition 2 (Goursat domain; McCuan 2018) Given $k \in \mathbb{N}\backslash\{1\}$, an open subset $\Omega \subset \mathbb{C}$ is a **Goursat domain of order** k if $\partial\Omega$ is a piecewise C^1 simple closed curve, and there exist k subdomains $\Omega_1, \Omega_2, \ldots, \Omega_k$ of Ω for which

(i) Each Ω_j is geometrically similar to Ω for $j = 1, 2, \ldots, k$, i.e., Ω_j has the form

$$\Omega_j = \{\rho(z) : z \in \Omega\}$$

where $\rho : \mathbb{C} \to \mathbb{C}$ is a composition of a dilation (complex scaling) and a translation.

(ii) Ω_j is geometrically congruent to Ω_ℓ for all ℓ and j, and

(iii) $\text{area}(\Omega) = \sum_{j=1}^{k} \text{area}(\Omega_j) = k\,\text{area}(\Omega_\ell)$ for each $\ell = 1, 2, \ldots, k$.

Exercise 5 Prove a version of Goursat's lemma which applies to any Goursat domain.

Exercise 6 Give examples of Goursat domains of all orders. Find all the "interesting" Goursat domains you can find.

Exercise 7 (presently considered very difficult; see *Geometria Dedicata* **82**, 325–344 (2000)) Classify all Goursat domains of order $k = 2$.

Exercise 8 (open problem) Give an "easy" proof of the classification of all Goursat domains of order $k = 2$.

Note: The study of Goursat domains as suggested above may be considered what is called in mathematics a "tiling problem." As a tiling problem, the subdomains Ω_j for $j = 1, 2, \ldots, k$ are said to "tile" the domain Ω. Among tiling problems, this kind of problem is said to involve repeated or replicated tiles. As a result, a reasonable name for such a tiling might be a "repeating tiling" or a "replication tiling," or even a "rep-tile." It seems, however, that the preponderance of mathematicians interested in these kinds of tiling problems have a predilection for cutesy names; in the literature such a tiling is referred to as a **reptile**.

4 Cauchy's theorem from Goursat's lemma

As an immediate consequence of Exercise 5 we obtain the following special case of Cauchy's theorem:

Theorem 3 (Cauchy's theorem for Goursat domains) If U is an open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ is a complex differentiable function, then

$$\int_\Gamma f = 0 \qquad \text{for every Goursat domain } \Omega \subset U \text{ with } \Gamma = \partial\Omega \subset U. \tag{7}$$

Our objective from this point is somewhat more modest than that suggested by Brown and Churchill. An explanation of why this is the case is given in the next section. In any case, our objective is, first of all, to **provide an approach** to obtain various Cauchy theorem**s** having basically the same form as Theorem 1 and Thereom 3 but applying to various classes of domains Ω. Here is a simple example of the kind of result we can obtain:

Theorem 4 (Cauchy's theorem for some semicircular domains) Let U be an open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ a complex differentiable function. For $z_0 \in \mathbb{C}$ and $r > 0$, consider a semicircular domain $\Omega \subset U$ having the form

$$\Omega = \{z \in \mathbb{C} : |z - z_0| < r \text{ and } \operatorname{Im}(z) > \operatorname{Im}(z_0)\}.$$

If $\Gamma = \partial\Omega \subset U$, then

$$\int_\Gamma f = 0. \tag{8}$$

Our first observation is that every triangular domain is a Goursat domain:[1]

Exercise 9 If Ω is an open subset of $\mathbb{C}$ and $\partial\Omega$ is a triangle, then Ω is a Goursat domain of order $k = 4$.

In particular, Theorem 3 applies to triangular domains. Now, let us take a particular special case of a semicircular domain (for simplicity):

$$\Omega = \{z \in \mathbb{C} : |z| < 1,\ \operatorname{Re}(z) > 0\}.$$

We are assuming $\overline{\Omega} \subset U$ where $f : U \to \mathbb{R}$ is complex differentiable.

[1]This approach may be found in the text of Stein and Shakarchi.

Because U is open, for each point $a \in \mathbb{R}$ with $|a| \leq 1$, there is some $\epsilon > 0$ so that the square domain

$$Q_x = \{z \in \mathbb{C} : |\operatorname{Re}(z) - x| < \epsilon_x \text{ and } |\operatorname{Im}(z)| < \epsilon_x\}$$

with center at x satisfies $Q_x \subset U$. Thus,

$$\{Q_x\}_{-1 \leq x \leq 1} \qquad \text{is a covering of the compact set } K = \{x \in \mathbb{R} : |x| \leq 1\}.$$

A **compact** set in $\mathbb{C}$ is a set which is closed and bounded. For our immediate purposes, it is important to know that if a compact set is "covered" by open sets, i.e., is the subset of the union of these open sets, then only finitely many of the open sets are needed to cover the compact set. Thus, only finitely many of the square domains still cover K. Let these square domains be centered at $x_1, x_2, \ldots, x_k$ and have "radii" $\epsilon_{x_1}, \epsilon_{x_2}, \ldots, \epsilon_{x_k}$. Let

$$\epsilon = \min\{\epsilon_{x_1}, \epsilon_{x_2}, \ldots, \epsilon_{x_k}\}.$$

Notice then that the line segment $L = \{x - i\epsilon : -1 \leq x \leq 1\}$ and the closed rectangle

$$R = \{x - ti : -1 \leq x \leq 1,\ -\epsilon \leq t \leq 0\}$$

are both entirely contained in U. In particular, for each $z \in \overline{\Omega}$, there is a unique path Z given by the **concatenation**[2] of Z_h and Z_v where Z_h is parameterized by ζ_h with

$$\zeta_h(t) = -1 - \epsilon i + t\operatorname{Re}(z), \qquad \text{for } 0 \leq t \leq 1$$

and Z_v is parameterized by ζ_v with

$$\zeta_v(t) = -1 + \operatorname{Re}(z) - (1-t)\epsilon i + t\operatorname{Im}(z), \qquad \text{for } 0 \leq t \leq 1.$$

Also, the path Z is entirely within U. Thus, we can define a function $g : \overline{\Omega} \to \mathbb{C}$ by

$$g(z) = \int_Z f.$$

We need to go a little bit farther than this, but let us pause to consider the value of g on the semicircular open region Ω. The basic claim is that $g : \Omega \to \mathbb{C}$ is complex differentiable with $g' = f$, that is, g is a complex antiderivative for f on Ω.

[2]We are departing here slightly from the convention that a curve is parameterized on a single interval, but hopefully the meaning is clear: The curve Z consists of a horizontal segment along the bottom boundary segment of $\partial\Omega$ and a vertical segment terminating at $z \in \Omega$.

If $z \in \Omega$ and we take $h + ik \in \mathbb{C}$ with $h, k \in \mathbb{R}$ and $|h + ik|$ small enough, then $z + h + ik$ will also be in Ω and we can consider the difference quotient

$$\frac{g(z + h + ik) - g(z)}{h + ik}.$$

Let us consider some special cases: If $h > 0$ and $k = 0$, then

$$g(z + h) - g(z) = \int_A f + \int_I f - \int_J f \tag{9}$$

where A, I, and J are segments as follows:

$$\begin{aligned}
&A: \ \gamma(t) = \mathrm{Re}(z) - \epsilon i + th \qquad \text{for } 0 \le t \le 1,\\
&I: \ \gamma(t) = \mathrm{Re}(z) + h - (1 - t)\epsilon i + t\,\mathrm{Im}(z)i \qquad \text{for } 0 \le t \le 1,\\
&J: \ \gamma(t) = \mathrm{Re}(z) - (1 - t)\epsilon i + t\,\mathrm{Im}(z)i \qquad \text{for } 0 \le t \le 1.
\end{aligned}$$

These are three sides of a rectangle (which is the boundary of a Goursat domain of order four in U). Specifically, if we take

$$B: \ z + th \qquad \text{for } 0 \le t \le 1,$$

then

$$\int_A f + \int_I f - \int_B f - \int_J f = 0,$$

or

$$g(z_h) - g(z) = \int_B f = \int_0^h f(z + t)\,dt.$$

Returning to the full difference quotient we have

$$\lim_{h \searrow 0} \frac{g(z + h) - g(z)}{h} = \lim_{h \searrow 0} \frac{1}{h} \int_0^h f(z + t)\,dt = f(z)$$

because f is continuous and the limiting quantity is the average of f on the segment B. Very explicitly using the basic estimates for complex integrals,

$$\left| \frac{1}{h} \int_0^h f(z + t)\,dt - f(z) \right| = \left| \frac{1}{h} \int_0^h [f(z + t) - f(z)]\,dt \right| \le \max_{0 \le t \le h} |f(z + h) - f(z)|.$$

Another special case: $h, k > 0$. In this case, (9) still holds with

$$\begin{aligned}
&A:\ \gamma(t) = \mathrm{Re}(z) - \epsilon i + th \qquad \text{for } 0 \le t \le 1,\\
&I:\ \gamma(t) = \mathrm{Re}(z) + h - (1-t)\epsilon i + t[\mathrm{Im}(z) + k]i \qquad \text{for } 0 \le t \le 1,\\
&J:\ \gamma(t) = \mathrm{Re}(z) - (1-t)\epsilon i + t\,\mathrm{Im}(z)i \qquad \text{for } 0 \le t \le 1.
\end{aligned}$$

Notice that only the segment I has changed. Furthermore, we can write I as the concatenation of I_1 and I_2 where

$$\begin{aligned}
&I_1:\ \gamma(t) = \mathrm{Re}(z) + h - (1-t)\epsilon i + t\,\mathrm{Im}(z)i \qquad \text{for } 0 \le t \le 1,\\
&I_2:\ \gamma(t) = z + h + tki \qquad \text{for } 0 \le t \le 1.
\end{aligned}$$

Using the same segment B from the case above, we see A, I_1, B, and J are segments along the boundary of a rectangle in U with

$$\int_A f + \int_{I_1} f - \int_B f - \int_J f = 0.$$

Introducing the segment C connecting z to $z + h + ik$, we see also that B and I_2 are two sides of a triangular domain (which is a Goursat domain of order 4) in U with

$$\int_B f + \int_{I_2} f - \int_C f = 0.$$

Combining these observations,

$$g(z + h + ik) - g(z) = \int_A f + \int_{I_1} f + \int_{I_2} f - \int_J f = \int_C f,$$

and

$$\begin{aligned}
\lim_{h,k \searrow 0} \frac{g(z+h) - g(z)}{h + ik} &= \lim_{h,k \searrow 0} \frac{1}{h+ik} \int_C f\\
&= \lim_{h,k \searrow 0} \frac{1}{h+ik} \int_0^1 f(z + t(h+ik))\,(h+ik)dt\\
&= f(z).
\end{aligned}$$

Exercise 10 Show carefully that for $z \in \Omega$

$$g'(z) = \lim_{h+ik \to 0} \frac{g(z+h+ik) - g(z)}{h+ik} = f(z).$$

In view of Exercise 10 we know that whenever Γ is a simple closed curve within Ω, then

$$\int_\Gamma f = 0$$

by the second special case of Cauchy's theorem in which we have a complex primitive.

The little bit more we need is to apply this argument to $\Gamma = \partial\Omega$. Let's see if we can get that. As we had for each point in $K = \mathbb{R} \cap \partial\Omega$, there is for each point $z \in P = \partial B_1(0) \cap \{w : \mathrm{Im}(w) \geq 0\}$ an open polar rectangle

$$R_z = \{te^{i\theta} : |t-1| < \delta_z, \ |\theta - \mathrm{Arg}(z)| < \delta_z\},$$

with "center" z and "radius" δ_z, satisfying $R_z \subset U$. Since P is also compact, we can take finitely many points $z_1, z_2, \ldots, z_m \in P$ and

$$\delta = \min\{\delta_{z_1}, \delta_{z_2}, \ldots, \delta_{z_m}\}$$

where

$$P \subset \bigcup_{j=1}^m R_{z_j}.$$

By taking ϵ and δ smaller if necessary, we can ensure both the square and the polar rectangles centered at $z = -1$ and $z = +1$ with radius $\rho = \min\{\epsilon, \delta\}$ are both in U. It follows that the (possibly new) horizontal segment

$$L_\rho = \{x - \rho i : |x| < 1 + \rho\}$$

and the entire semicircular domain

$$\Sigma = \{z \in \mathbb{C} : |z - \rho i| < 1 + \rho, \ \mathrm{Im}(z) > -\rho\}$$

are entirely contained in U. Letting Z denote, for each $z \in \Sigma$ the concatenation of a horizontal segment connecting $-1-\rho-i\rho$ to $\mathrm{Re}(z) - i\rho$ followed by the vertical segment connecting $\mathrm{Re}(z) - i\rho$ to z, we obtain $g : \Sigma \to \mathbb{C}$ by

$$g(z) = \int_Z f$$

which is complex differentiable on Σ, an open set containing $\overline{\Omega}$, and is a complex antiderivative of f: $g' = f$. $\square$

It would be nice to have a general result for domains like the semicircular domain above and the Stein domains of type 1 defined below giving a little bit larger open set Σ with

$$\overline{\Omega} \subset \Sigma \subset \overline{\Sigma} \subset U$$

and such that Σ also preserves the geometry of Ω to the extent that an antiderivative may be defined on some open set containing $\overline{\Omega}$ using concatenated segments from some "lower left" point as above. I have not thought carefully about how to formulate or prove such a result, but I will give you a chance to do that:

Definition 3 Given bounded open sets U and Ω in $\mathbb{C}$ with $\partial\Omega$ a simple closed curve satisfying

$$\partial\Omega \subset \overline{\Omega} \subset U,$$

we say Ω is a **Stein domain of type 1** if for some $y_0, a, b \in \mathbb{R}$ with $a < b$ the horizontal line segment $L = \{x + y_0 i : a \le x \le b\}$ satisfies $L \subset U$, and for each $z \in \overline{\Omega}$ the segment

$$V = \{\mathrm{Re}(z) + (1-t)y_0 i + t\,\mathrm{Im}(z) : 0 \le t \le 1\}$$

satisfies $V \subset U$.

Exercise 11 State and prove a version of Cauchy's theorem which applies to every Stein domain of type 1.

Stein generalizes this argument to apply to, for example, a disk $B_r(z_0)$ which is not a Stein domain of type 1. The generalization proceeds as follows: Given $z \in B_r(z_0)$, there is a unique path Γ connecting z_0 to z which is the concatenation of a horizontal segment followed by a vertical segment. There is some $\epsilon > 0$ for which $B_\epsilon(\Omega) = B_{r+\epsilon}(z_0) \subset U$ when as usual $f : U \to \mathbb{C}$, and the function $g : B_{r+\epsilon}(z_0) \to \mathbb{C}$ by

$$g(z) = \int_\Gamma f$$

is a complex differentiable antiderivative for f on $B_\epsilon(\Omega)$. Since the horizontal and vertical segments in this case are not always "to the right" and "up" respectively in this situation, consideration of the difference quotient falls also into several cases which is a little tedious, but it works in the end.

Definition 4 Given bounded open sets U and Ω in $\mathbb{C}$ with $\partial\Omega$ a simple closed curve satisfying

$$\partial\Omega \subset \overline{\Omega} \subset U,$$

we say Ω is a **Stein domain of type 2** if for some $z_0 \in \Omega$ the horizontal line segments $L = \{z_0 + t[\mathrm{Re}(z) - \mathrm{Re}(z_0)] : 0 \le t \le 1\}$ and $V = \{\mathrm{Re}(z) + i\,\mathrm{Im}(z_0) + t[\mathrm{Im}(z) - \mathrm{Im}(z_0)] : 0 \le t \le 1\}$ satisfy $L, V \subset U$.

Exercise 12 State and prove a version of Cauchy's theorem which applies to every Stein domain of type 2.

There should be a collection of **Stein domains of type 3** which are essentially what Stein calls "simple domains" or "elementary domains" or something like that. I started to formulate and prove a Cauchy theorem for such domains, but I never did quite work out the details. I don't feel too bad because Stein did not seem to work out the details either. There are only so many hours in a life.

5 Brown and Churchill's proof

It seems to me there is probably something of an error in the proof of Cauchy's theorem given in Brown and Churchill, though the basic assertion(s) may be correct. In order to describe one specific point that troubles me, let's specialize the proof to the special case of, for example, a square domain:

$$\Omega = \{z \in \mathbb{C} : |\,\mathrm{Re}(z)|, |\,\mathrm{Im}(z)| < 1\}.$$

They essentially seem to be following the original proof of Goursat and claim something like this:

Lemma 2 (Goursat's lemma) For any $\epsilon > 0$, there is a partition of Ω into finitely many subsquares $Q_1, Q_2, \ldots, Q_k$ such that there exists, in each subsqure Q_j for $j = 1, 2, \ldots, k$, a point ζ_j such that

$$|f(z) - f(\zeta_j) - f'(\zeta_j)(z - \zeta_j)| \le \epsilon |z - \zeta_j|$$

for every $z \in Q_j$.

Perhaps the simplest thing to say is that I don't see how to prove this result, and I don't think Brown and Churchill (and probably also Goursat) have a correct proof.

Here is an interesting story: There is a famous result called the Cantor-Bernstein theorem. Bernstein was very interested in the result and went around giving lectures on it and popularizing it. Bernstein had published a "proof" of the result and was presenting that "proof" in his lectures. Bernstein's proof was incorrect, and as far as

I can tell, Bernstein died before a correct proof was given. I think maybe he went to his grave thinking he had a correct proof. In any case, he didn't give a correct proof, but he got his name on the result anyway.

It seems that maybe something similar is the case with Goursat's theorem relaxing the requirement that f' is continuous in Cauchy's integral theorem. Goursat was very interested in this result. He published a "proof" of the result, and he got his name on the result. But I'm not entirely sure he had a correct proof of the result.

It seems like E.H. Moore tried to clean this up and give a correct proof, which is essentially Stein's proof that I've presented above, without using Goursat's lemma. It may be a little disappointing that we have not obtained Cauchy's theorem in the generality stated at the very beginning (general simple closed curves) but only for domains with some special geometry like that of the semicircular domain or like that of the disk, or maybe for Stein domains of type 3, whatever those are. On the other hand, Moore doesn't really get that generality either. If you look closely, he has some pretty complicated conditions on the curves to which he claims his argument will apply. In practice, one would have to take a specific domain (like a semicircular domain, a disk, or a Stein domain of type 3 or whatever) and check those complicated conditions.

My conclusion is that what we've got is, on the one hand, pretty good, and on the other hand what we've got may not be too much weaker than the best that has actually been carefully proved, e.g., by E.H. Moore.

One final thing to note is that I'm pretty sure Brown and Churchill restrict to piecewise C^1 curves. (This is not a restriction used by Moore.) It is an interesting fact, that if you start partitioning a piecewise C^1 curve using coordinate squares or rectangles, then you can quite easily end up with a (sub)region whose boundary is **not** piecewise C^1 anymore. Roughly speaking, Moore uses rectifiability,[3] and it's pretty clear that cutting up a domain with rectifiable boundary using rectangles leads to domains with rectifiable boundaries, but this is definitely a detail that should be checked.

[3]Moore also needs to be able to integrate on these curves, and I guess rectifiability of a continuous curve is enough, but one needs a better notion of the complex integral than the one we have for that. We basically need something like piecewise C^1 for our complex integrals.