Assignment 8: Elementary functions
Chapter 3 of BC
Due Wednesday, March 15, 2023

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Problem 1 (complex exponential) Draw the following subsets of the complex z-plane and their images under the exponential function in the specified complex w-plane/sheet. Use colors.

(a) $\Sigma_0 = \{x + iy : 0 \le y < 2\pi\}$ in $\mathcal{L}_0 = \{w \in \mathbb{C}\backslash\{0\} : 0 \le \arg(w) < 2\pi\}$.

(b) $\{iy : 2\pi \le y < 4\pi\}$ in $\mathcal{L}_1 = \{w \in \mathbb{C}\backslash\{0\} : 2\pi \le \arg(w) < 4\pi\}$.

(c) $\{2 + iy : 2\pi \le y < 5\pi\}$ in $\mathcal{L}_1 \cup \mathcal{L}_2$ where $\mathcal{L}_1 = \{w \in \mathbb{C}\backslash\{0\} : 2\pi \le \arg(w) < 4\pi\}$ and $\mathcal{L}_2 = \{w \in \mathbb{C}\backslash\{0\} : 4\pi \le \arg(w) < 6\pi\}$.

(c) $\{-\ln(2) + iy : -5\pi \le y < \pi\}$ in the appropriate sheets of the Riemann surface $\mathcal{L}$ for the complex exponential (which you should define).

(10 presentation points)

Problem 2 (complex logarithm) Explain why

$$\log_0(w) = \ln|w| + i\tan^{-1}\left(\frac{\text{Im}(w)}{\text{Re}(w)}\right) \tag{1}$$

for $w \in \{\zeta \in \mathcal{L}_0 : \text{Re}(\zeta), \text{Im}(\zeta) > 0\}$ where $\mathcal{L}_0$ is given in Problem 1 above and $\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$ denotes the real (principal) arctangent. (10 presentation points)

Problem 3 (complex logarithm) Find generalized versions of the formula (1) giving the real and imaginary parts of $\log_0(w)$ in terms of the real (principle) arctangent for all $w \in \mathcal{L}_0$. (10 presentation points)

Problem 4 (complex logarithm) In view of the formula (1) what interesting thing can you assert/say about the function

$$v(x, y) = \ln\left(\frac{y}{x}\right)?$$

I'm leaving it to you to properly specify the domain of the function v. (10 presentation points)

Problem 5 (complex exponential/logarithm) Write down the set description of the j-th sheet of the Riemann surface $\mathcal{L}_j$ associated with the complex exponential. (10 presentation points)

Problem 6 (Exercise 3.30.1 in BC) Calculate

$$\exp\left(\frac{2 + \pi i}{4}\right).$$

(10 presentation points)

Problem 7 (Exercise 3.30.3 in BC) Show the composition $f(z) = \exp(\overline{z})$ is nowhere complex differentiable. Hint: Cauchy-Riemann equations.
(10 presentation points)

Problem 8 (Exercise 3.1 in my notes and Exercise 3.30.4 in BC)

(a) Show the complex exponential function is entire.

(b) Compute the derivative of $\exp(z)$.

(c) Show the composition $f(z) = \exp(z^2)$ is entire and compute the derivative of f.

(10 presentation points)

Problem 9 (Exercises 3.30.5 in BC)

(a) With $z = x + iy$ express $|\exp(2z + i)|$ and $|\exp(iz^2)|$ as functions of x and y.

(b) Show

$$\left|e^{2z+i} + e^{iz^2}\right| \leq e^{2\,\mathrm{Re}\,z} + e^{-2\,\mathrm{Re}(z)\,\mathrm{Im}(z)}. \tag{2}$$

(c) Characterize the condition of equality in (2).

(10 presentation points)

Problem 10 (Exercise 3.30.6 in BC)

(a) Show

$$\left|e^{z^2}\right| \leq e^{|z|^2}. \tag{3}$$

(b) Characterize the condition of equality in (3).

(10 presentation points)