Assignment 7: Complex Differentiability
Due Wednesday, March 8, 2023

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Problem 1 (Exercise 2.24.1-2 in BC) Find the real and imaginary parts u and v of the following functions $f : U \to \mathbb{C}$ and calculate the first (real) partial derivatives

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \text{and} \quad \frac{\partial v}{\partial y}$$

at any points where they exist. Identify a "natural" domain $U \subset \mathbb{C}$ for each function and determine of the Cauchy-Riemann equations are satisfied on $\gamma^{-1}(U)$.

(a) $f(z) = z - \overline{z}$.

(b) $f(z) = e^x \, e^{-iy}$.

(c) $f(z) = e^{-x} \, e^{-iy}$.

(d) $f(z) = \cos x \, \cosh y - i \, \sin x \, \sinh y$.

(10 presentation points)

Problem 2 (Exercise 2.24.2 in BC) Determine which (if any) of the functions $f : U \to \mathbb{C}$ given in Problem 1 above are differentiable on an open set $U \subset \mathbb{C}$ and, if so, find the second and third complex derivatives f'' and f''' of these functions. (10 presentation points)

Problem 3 (Exercise 2.24.3 in BC) Consider the function $f : \mathbb{C} \to \mathbb{C}$ with values given by $f(z) = x^2 + iy^2$. Determine all points $z = x + iy \in \mathbb{C}$ where f is differentiable and calculate $f'(z)$ at those points. (10 presentation points)

Problem 4 (Exercise 2.21 in my notes) Let $u : \Omega \to \mathbb{R}$ be a real valued function defined on an open subset $\Omega \subset \mathbb{R}^2$. Assume u has continuous first partial derivatives on Ω, i.e., $u \in C^1(\Omega)$. Writing

$$u(x+h, y+k) - u(x, y) = u(x+h, y+k) - u(x+h, y) + u(x+h, y) - u(x, y)$$

estimate the difference as suggested in the following steps:

(a) Draw the points (x, y), $(x+h, y)$, and $(x+h, y+k)$ in $\mathbb{R}^2$ for representative values of x, y, h, and k.

(b) Show that for some $\delta_0 > 0$

$$\{(x+t, y) : |t| < |h|\} \subset \Omega \qquad \text{and} \qquad \{(x+h, y+t) : |t| < |k|\} \subset \Omega$$

whenever $h^2 + k^2 < \delta_0^2$.

(c) Draw a second picture in $\mathbb{R}^2$ to illustrate the assertion of part **(b)** above.

(d) Taking $h^2 + k^2 < \delta_0^2$, define a function $g : [-|h|, |h|] \to \mathbb{R}$ satisfying

(i) $g(0) = u(x, y)$,

(ii) $g(h) = u(x+h, y)$, and

(iii)

$$g'(t) = \frac{\partial u}{\partial x}(x+t, y).$$

(e) Apply the mean value theorem to g to express the difference $u(x+h, y) - u(x, y)$ in terms of $\partial u/\partial x$.

(f) Let $N = \{(x, -x) : x \in \mathbb{R}\}$. Find

$$\lim_{\mathbb{R}^2 \backslash N \ni (h,k) \to (0,0)} \frac{1}{h+k}[u(x+h, y+k) - u(x, y)].$$

(10 presentation points)

Problem 5 (little-o and differentiability for functions of two variables) Under the assumption $u \in C^2(\Omega)$ as described in Problem 4 above, Brown and Churchill claim (page 66, Chapter 2, Section 23) that for any $(x_0, y_0) \in \Omega$

$$u(x_0+h, y_0+k) - u(x_0, y_0) = \frac{\partial u}{\partial x}(x_0, y_0)\ h + \frac{\partial u}{\partial x}(x_0, y_0)\ k + h\phi_1(h, k) + k\phi_2(h, k) \quad (1)$$

where ϕ_1 and ϕ_2 satisfy

$$\lim_{(h,k)\to(0,0)} \phi_j(h, k) = 0 \qquad \text{for } j = 1, 2. \quad (2)$$

Functions satisfying the limiting assertions of (2) are said to be "little-o of one as (h, k) tends to $(0, 0)$," that is, when you divide the quantity by 1, the result tends to zero as (h, k) tends to $(0, 0)$. One writes in this case, $\phi_j = \circ(1)$. One can extend the notion of "little-o" and its accompanying notation to division by other quantities as well. Thus (1) may also be written as

$$u(x_0+h, y_0+k) - u(x_0, y_0) = \frac{\partial u}{\partial x}(x_0, y_0)\ h + \frac{\partial u}{\partial x}(x_0, y_0)\ k + \circ(h) + \circ(k) \qquad \text{as } (h, k) \to (0, 0).$$

Finally, I mention in passing that (1) is essentially the *definition* of **differentiability** for a function u of two variables. More precisely, $u : \Omega \to \mathbb{R}$ is differentiable at $(x_0, y_0) \in \Omega$ if there is some linear function $L : \mathbb{R}^2 \to \mathbb{R}$ for which

$$u(x_0 + h, y_0 + k) - u(x_0, y_0) = L(h, k) + \circ(h) + \circ(k) \qquad \text{as } (h, k) \to (0, 0).$$

The linear function turns out to be given as a dot product with the vector of first partial derivatives, i.e., the gradient. You may not have seen, or noticed or thought about, this definition before. (If that's the case, you now have a chance!) The assertion here, it will be noted, is that C^2 implies differentiability.
(10 presentation points)

Problem 6 (polar coordinates) Let $f : U \to \mathbb{C}$ be a complex differentiable function defined on an open set $U \subset \mathbb{C}$. We write the real and imaginary parts of f as $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ as functions of $(x, y) \in \mathbb{R}^2$ as usual. Define $\alpha : \Omega \to \mathbb{R}$ and $\beta : \Omega \to \mathbb{R}$ by

$$\alpha(r, \theta) = u(r\cos\theta, r\sin\theta) \qquad \text{and} \qquad \beta(r, \theta) = v(r\cos\theta, r\sin\theta).$$

These are the real and imaginary parts of f in polar coordinates.

(a) Recall that the domain of u and v is $\gamma^{-1}(U)$ where $\gamma : \mathbb{R}^2 \to \mathbb{C}$ is the canonical identification. What can you say about the domain of α and β?

(b) Recall that u and v satisfy the Cauchy-Riemann equations. Show that α and β satisfy the system of first order partial differential equations

$$\begin{cases} \dfrac{\partial \alpha}{\partial r} &= \dfrac{1}{r}\dfrac{\partial \beta}{\partial \theta} \\[2ex] \dfrac{\partial \alpha}{\partial \theta} &= -r\dfrac{\partial \beta}{\partial r}. \end{cases} \tag{3}$$

These are called the **Cauchy-Riemann equations in polar coordinates**.

(10 presentation points)

Problem 7 (polar coordinates) What can you say about the division by r in the first PDE of (3)?
(10 presentation points)

Problem 8 (polar coordinates, Problem 6 above) Use the theorem concerning sufficiency of the Cauchy-Riemann equations to state and prove a theorem about the sufficiency of the Cauchy-Riemann equations in polar coordinates. (See Theorem 3 of Chapter 2 in my notes or the theorem on page 66 of Chapter 2 Section 23 in BC.)
(10 presentation points)

Problem 9 (polar coordinates, Exercises 2.24.6-8 in BC, cf. Problem 9 Assignment 6) Let $f : U \to \mathbb{C}$ be a complex differentiable function on an open set $U \subset \mathbb{C}$. Assume there is an open set $Q \subset \mathbb{R}^2 \backslash \{(0, 0)\}$ and there are functions $\alpha, \beta : Q \to \mathbb{R}$ for which

$$\alpha(r, \theta) + i\beta(r, \theta) = f(re^{i\theta}) \qquad \text{for} \qquad (r, \theta) \in Q.$$

Given $(r_0, \theta_0) \in Q$, obtain formula(s) for

$$f'(r_0 e^{i\theta_0}) = \lim_{\zeta \to r_0 e^{i\theta_0}} \frac{f(\zeta) - f(r_0 e^{i\theta_0})}{\zeta - r_0 e^{i\theta_0}}$$

in terms of the partial derivatives of $\alpha = \mathrm{Re}(f)$ and $\beta = \mathrm{Im}(f)$.
(10 presentation points)

Problem 10 (Exercise 2.26.6 in BC) Consider the slit plane

$$U = \{re^{i\theta} : 0 < r,\ 0 < \theta < 2\pi\} \subset \mathbb{C}$$

and the corresponding strip

$$Q = \{(r, \theta) : 0 < r,\ 0 < \theta < 2\pi\} \subset \mathbb{R}^2.$$

(i) Show the polar coordinates map $\psi : Q \to U$ by

$$\psi(r, \theta) = re^{i\theta}$$

is a smooth bijection in the sense that

$$\mathrm{Re}(\psi) = r\cos\theta,\ \mathrm{Im}(\psi) = r\sin\theta \in C^\infty(Q)$$

and $\psi^{-1} \circ \gamma^{-1} = (\rho, t) = (\|(x, y)\|, \mathrm{Arg}(\gamma(x, y))) : \gamma^{-1}(U) \to Q$ has $\rho, t \in C^\infty(\gamma^{-1}(U))$.

(ii) Define $f : U \to \mathbb{C}$ by

$$f(z) = \ln|z| + i\,\mathrm{Arg}(z) = \alpha + i\beta$$

where $\mathrm{Arg}(z) \in (0, 2\pi)$ and $\alpha, \beta : Q \to \mathbb{R}$ by

$$\alpha(r, \theta) = \ln r \qquad \text{and} \qquad \beta(r, \theta) = \theta.$$

Use the result of Problem 8 above or the result of Problem 9 above to show f is complex differentiable.

(iii) Use the result of Problem 9 above to show

$$f'(z) = \frac{1}{z}.$$

Note: The function f defined here is essentially a branch of the **complex logarithm**.
(10 presentation points)