Assignment 6 = Exam 2: Limits and Differentiability Due Wednesday, March 1, 2023

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Problem 1 (Exercise 2.18.10 in BC) Calculate the limits:

(a)

$$\lim_{z\to\infty} \frac{4z^2}{(z-1)^2}.$$

(b)

$$\lim_{z\to 1} \frac{1}{(z-1)^3}.$$

(c)

$$\lim_{z\to\infty} \frac{z^2+1}{z-1}.$$

(10 presentation points)

Problem 2 (Exercise 2.20.5 in BC) Given an open set $U \subset \mathbb{C}$ and functions $f, g : U \to \mathbb{C}$ both differentiable on U, show

$$\frac{d}{dz}(f+g) = f' + g'.$$

(10 presentation points)

Problem 3 (L'Hopital's rule; Exercise 2.20.4 in BC) Given an open set $U \subset \mathbb{C}$ and functions $f, g : U \to \mathbb{C}$ both differentiable on U and satisfying $f(z_0) = g(z_0) = 0$ and $g'(z_0) \neq 0$ for some $z_0 \in U$, show the limit

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} \qquad \text{exists}$$

and satisfies

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

(10 presentation points)

Problem 4 (Exercise 2.20.8 in BC) Determine all points of differentiability for these functions defined on all of $\mathbb{C}$:

(a) $f(z) = \overline{z}$.

(b) $f(z) = \mathrm{Re}(z)$.

(c) $f(z) = \mathrm{Im}(z)$.

(d) $f(z) = |z|^2$.

(10 presentation points)

Problem 5 (Exercise 2.20.9 in BC) Determine all points of differentiability of the function $f : \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = \begin{cases} |z|^2 \dfrac{\overline{z}}{z^2}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

(10 presentation points)

Problem 6 Let $f : U \to \mathbb{C}$ be a complex differentiable function defined on an open set $U \subset \mathbb{C}$, and let $z_0 \in U$ be fixed. Consider the function $g : U \to \mathbb{C}$ given by

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0), & z \neq z_0 \\ 0, & z = z_0. \end{cases}$$

Show g is continuous on U.
(10 presentation points)

Problem 7 (Exercise 1.20.3 in BC) Show that any polynomial

$$P(z) = \sum_{j=0}^{n} a_j z^j$$

of degree n has coefficients satisfying

$$a_j = \frac{P^{(j)}(0)}{j!} \qquad \text{for } j = 0, 1, 2, \ldots, n.$$

(10 presentation points)

Problem 8 (Exercise 2.16 from my notes; Exercise 2.20.10 in BC) The real Legendre polynomials can be defined as follows $P_0(x) \equiv 1$, and for each $n = 1, 2, 3, \ldots$, P_n is the degree n polynomial for which the following hold

(i) $P_n(1) = 1$.

(ii)

$$\int_{-1}^{1} P_n(x) P_j(x)\, dx = 0 \qquad \text{for } 0 \leq j < n.$$

(a) Plot the first few (real) Legendre polynomials on the interval $[-1, 1]$.

(b) Show the real Legendre polynomials are given by

$$P_n(x) = \frac{1}{n! 2^n} \frac{d^n}{dx^n} (x^2 - 1)^n \qquad \text{for } n = 1, 2, 3, \ldots.$$

The complex Legendre polynomials $P_n = P_n(z)$ have (and satisfy) the same formulas as polynomial functions on the complex plane.
(10 presentation points)

Problem 9 (Exercise 2.17 in my notes) Let $f : U \to \mathbb{C}$ be a complex differentiable function on an open set $U \subset \mathbb{C}$. Given $z \in U$, calculate the limit of the difference quotient

$$f'(z) = \lim_{\zeta \to z} \frac{f(\zeta) - f(z)}{\zeta - z}$$

using values $\zeta = z + ih$ where $h \in \mathbb{R}$. Obtain a formula for $f'(z)$ in terms of the partial derivatives of $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$.

Calculate the same limit using values $\zeta = z + h$ for $h \in \mathbb{R}$.
(10 presentation points)

Problem 10 (Laplace's equation: the equation of harmonic functions) Let $u, v : \Omega \to \mathbb{R}$ be two real valued functions defined on an open set $\Omega \subset \mathbb{R}^2$ satisfying the following:

(i) All the partial derivatives of u and v up to second order exist and are continuous at all points of Ω.

(ii) The functions u and v satisfy the **Cauchy-Riemann equations**:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Show u and v (both) satisfy Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

(10 presentation points)