Assignment 5: Limits and Differentiability
Due Wednesday, February 22, 2023

John McCuan

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Problem 1 (Exercise 2.9 in my notes) We have defined a **limit**

$$\lim_{\zeta \to z} f(z) = w \in \mathbb{C}$$

where $f : U \backslash \{z\} \to \mathbb{C}$ and U is an open subset of $\mathbb{C}$ as follows:

For any $\epsilon > 0$, there is some $\delta > 0$ for which

$$0 < |\zeta - z| < \delta \qquad \Longrightarrow \qquad |f(\zeta) - w| < \epsilon.$$

Show that if

$$\lim_{z \to z_0} g(z) \qquad \text{and} \qquad \lim_{z \to z_0} h(z) \qquad \text{exist,}$$

then

(a)

$$\lim_{z \to z_0} [g(z) + h(z)] = \lim_{z \to z_0} g(z) + \lim_{z \to z_0} h(z), \qquad \text{and}$$

(b)

$$\lim_{z \to z_0} [g(z)h(z)] = \left(\lim_{z \to z_0} g(z) \right) \left(\lim_{z \to z_0} h(z) \right).$$

(10 presentation points)

Problem 2 (Exercise 2.4 in my notes) We can generalize the definition of a limit given in Problem 1 above as follows: Given $g : V \to \mathbb{C}$ with V an open subset of $\mathbb{C}$ and $z \in \overline{V}$, we say

$$\lim_{\zeta \to z} g(z) = w \in \mathbb{C}$$

if the following condition holds:

For any $\epsilon > 0$, there is some $\delta > 0$ for which

$$\left.\begin{array}{c} 0 < |\zeta - z| < \delta \\ \zeta \in V \end{array}\right\} \qquad \Longrightarrow \qquad |g(\zeta) - w| < \epsilon.$$

(a) Show the new definition is a **proper generalization** of the first definition. This means that in every case where the first definition may be applied, the second definition may also be applied and gives the same concept (of limit).

(b) Consider $V = \{w \in \mathbb{C}\backslash\{0\} : -\pi < \arg(w) < \pi\}$. Find ∂V.

(c) Let $g : V \to \mathbb{C}$ be a branch of $\sqrt{w}$ defined on the open set V from part **(a)** above. If $w_0 \in \partial V$, what can you say about

$$\lim_{w \to w_0} g(w)?$$

(10 presentation points)

Problem 3 (Exercise 2.4 in my notes) Show that the limit of a function

$$\lim_{\zeta \to z} f(\zeta) = w \in \mathbb{C}$$

with $\zeta \in \partial U$ as defined in Problem 2 above is **unique**. (10 presentation points)

Problem 4 (Exercise 2.11 in my notes) Let U be an open subset of $\mathbb{C}$ with $z_0 = x_0 + iy_0 \in U$, and assume $f : U\backslash\{z_0\} \to \mathbb{C}$ satisfies

$$\lim_{z \to z_0} f(z) = w \in \mathbb{C}.$$

Then show

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = \operatorname{Re} w \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = \operatorname{Im} w$$

where $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ (as usual). (10 presentation points)

Problem 5 (Exercise 2.12 in my notes; Exercises 2.18.1-4 in BC) Consider complex conjugation $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = \overline{z}$.

(a) Show g is continuous on all of $\mathbb{C}$.

(b) Show g is not differentiable at any single point $z \in \mathbb{C}$.

(10 presentation points)

Problem 6 (Example 2.13 in my notes) Show that if U is an open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ is differentiable, then the functions $u = \operatorname{Re} f : \gamma^{-1}(U) \to \mathbb{R}$ and $v = \operatorname{Im} f : \gamma^{-1}(U) \to \mathbb{R}$ where $\gamma : \mathbb{R}^2 \to \mathbb{C}$ is the canonical isomorphism from $\mathbb{R}^2$ to $\mathbb{C}$ have well-defined first partial derivatives at every point $(x, y) \in \gamma^{-1}(U) \subset \mathbb{R}^2$.
(10 presentation points)

Problem 7 (Exercise 2.14 in my notes) Give an example of a function $u : \mathbb{R}^2 \to \mathbb{R}$ for which both first partial derivatives exist at every point $(x, y) \in \mathbb{R}^2$, but u is not continuous at (at least) one point in $\mathbb{R}^2$. (10 presentation points)

Problem 8 Let $f = u + iv : U \to \mathbb{C}$ be a complex valued function of one complex variable z in an open set $U \subset \mathbb{C}$. Show the following:

(a) If f is continuous, then $u : \gamma^{-1}(U) \to \mathbb{R}$ and $v : \gamma^{-1}(U) \to \mathbb{R}$ are continuous.

(b) If f is differentiable, then f is continuous.

(10 presentation points)

Problem 9 (Exercise 2.18.2 in BC) Show that if $a, b, z_0 \in \mathbb{C}$, then

$$\lim_{z \to z_0} (az + b) = az_0 + b,$$

and describe the continuous mapping $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = az + b$. (10 presentation points)

Problem 10 (Exercise 2.15 in my notes) Let U be an open subset of $\mathbb{C}$ for which there is some $R > 0$ with $\mathbb{C} \backslash B_R(0) \subset U$. Indicating the value of R in your drawings, illustrate and explain the following:

(a) U as a **neighboord of ∞ in** $\mathbb{C}$.

(b) U as a **punctured neighborhood of** ∞ in the extended complex plane $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

(c) $U \cup \{\infty\}$ as a **neighborood of** ∞ **in** $\mathbb{C}_\infty$.

(10 presentation points)

Problem 11 (stereographic projection) Let $\gamma : \mathbb{R}^2 \to \mathbb{C}$ denote the canonical isomorphism, and let $\sigma : \mathbb{S}^2 \to \mathbb{C} \cup \{\infty\}$ denote **stereographic projection** given by

$$\sigma(x, y, z) = \begin{cases} \infty, & \text{if } z = 1 \\ \dfrac{x}{1-z} + i\dfrac{y}{1-z}, & \text{if } z < 1 \end{cases}$$

where $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is the **two-sphere**.

(a) Consider the map $\sigma_0 : \mathbb{S}^2 \backslash \{(0, 0, 1)\} \to \mathbb{R}^2$ given by

$$\sigma_0(x, y, z) = t(x, y)$$

where t satisfies

$$(1 - t)(0, 0, 1) + t(x, y, z) = t(x, y, 0). \tag{1}$$

(i) Explain what is happening here geometrically with pictures, and

(ii) show analytically that (1) has a unique solution t, and

(iii) $\sigma(x, y, z) = \gamma \circ \sigma_0(x, y, z)$ for $(x, y, z) \in \mathbb{S}^2 \backslash \{(0, 0, 1)\}$.

(b) Show σ is a bijection and find a/the formula for $\sigma^{-1} : \mathbb{C} \cup \{\infty\} \to \mathbb{S}^2$.

(10 presentation points)
Both $\mathbb{C} \cup \{\infty\}$ and $\mathbb{S}^2$ are called the **Riemann sphere** in this context.