Assignment 4: Functions
Due Wednesday, February 15, 2023

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Problem 1 (Exercises 2.2-3 in my notes) Show that a function $f : X \to Y$ is one-to-one and onto if and only if there exists a function $g : Y \to X$ satisfying

$$g \circ f = \mathrm{id}_X \qquad \text{and} \qquad f \circ g = \mathrm{id}_Y.$$

(10 presentation points)

Problem 2 (Exercise 2.4 in my notes) Consider $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ by

$$f(z) = \frac{1}{z}.$$

(a) Writing $z = x + iy$ find the real and imaginary $u : \mathbb{R}^2\backslash\{(0,0)\} \to \mathbb{R}$ and $v : \mathbb{R}^2\backslash\{(0,0\} \to \mathbb{R}$ parts of f.

(b) Compute the partial derivatives:

$$\frac{\partial u}{\partial x}, \ \frac{\partial u}{\partial y}, \ \frac{\partial v}{\partial x}, \ \text{and} \ \frac{\partial v}{\partial y}.$$

(c) Compute the second homogeneous partial derivatives:

$$\frac{\partial^2 u}{\partial x^2}, \ \frac{\partial^2 u}{\partial y^2}, \ \frac{\partial^2 v}{\partial x^2}, \ \text{and} \ \frac{\partial^2 v}{\partial y^2}.$$

(10 presentation points)

Problem 3 (Exercise 2.4 in my notes) Consider $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ by

$$f(z) = \frac{1}{z}$$

as in Problem 2 above.

(a) Draw the graphs of $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$.

(b) Draw a mapping picture for f based on the **orthogonal net** of circles and lines in the domain $\mathbb{C}\backslash\{0\}$ determined by

$$\{re^{i\theta} : \theta \in \mathbb{R}\} \qquad \text{and} \qquad \{re^{i\theta} : r > 0\}.$$

(10 presentation points)

Problem 4 (Example 2.13.5 in BC) Brown and Churchill assert that $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \sqrt{|z|}e^{i\,\mathrm{Arg}(z)/2}$$

where $-\pi < \mathrm{Arg}(z) \leq \pi$ is a well-defined single valued function on all of $\mathbb{C}$. This is correct, but this function is not (even) **continuous**. Explain why. (10 presentation points)

Problem 5 (Exercise 2.14.5 in BC) Consider the complex polynomial function determined by $f(z) = z^2$ and the (co)domain

$$V = \{u + iv : 1 < u < 2 \text{ and } 1 < v < 2\} \subset \mathbb{C}.$$

Find an open set $U \subset \mathbb{C}$ such that the restriction

$$f\big|_U : U \to V$$

is a bijection. (10 presentation points)

Problem 6 Find polynomial functions $P : \mathbb{C} \to \mathbb{C}$ which correspond to the following bijections of the (complex) plane:

(a) Translation to the right by $c > 0$.

(b) Rotation clockwise by an angle $\theta > 0$.

Show that if $P : \mathbb{C} \to \mathbb{C}$ given by

$$P(z) = \sum_{j=1}^{k} a_j z^j$$

is a polynomial function with $k \geq 2$ and $a_k \neq 0$, then P does not give a bijection of $\mathbb{C}$. (10 presentation points)

Problem 7 (complex squares) Draw the following subsets of $\mathbb{C}$:

(a) $\{z^2 \in B_1(0) : \pi/6 < \arg(z) < 5\pi/6\}$.

(b) $\{z^2 \in B_1(0) : -\pi/3 < \arg(z) < \pi/3\}$.

(10 presentation points)

Problem 8 (complex squares) Let $x_0 > 0$ and consider

$$U = \{x_0 + iy \in \mathbb{C} : y \in \mathbb{R}\}$$

and let $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z^2$.

(a) Draw $f(U)$.

(b) Express $u = \mathrm{Re}(f)$ on ∂U as a function of y.

(c) Express $u = \mathrm{Re}(f)$ on $\partial f(U)$ as a function of $v = \mathrm{Im}(f)$.

(d) Identify $\partial f(U)$ as a well-known curve.

(e) Describe $\partial f(U)$ as a function of x_0.

(10 presentation points)

Problem 9 (Complex square roots) Let $g : \{w \in \mathbb{C} : -\pi < \arg(w) \leq \pi\}$ be a branch of $\sqrt{w}$ and for $u_0 > 0$ fixed consider

$$V = \{u_0 + iv \in \mathbb{C} : v \in \mathbb{R}\}.$$

(a) Draw $g(V)$.

(b) Express $x = \mathrm{Re}(g)$ as a function of $y = \mathrm{Im}(g)$ on $\partial g(V)$.

(c) Identify $\partial g(V)$ as a well-known curve.

(e) Describe $\partial g(V)$ as a function of u_0.

(10 presentation points)

Problem 10 Consider the function $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = e^z$ using the definition

$$e^z = e^{\mathrm{Re}(z)}[\cos \mathrm{Im}(z) + i \sin \mathrm{Im}(z)].$$

Find a connected set $A \subset \mathbb{C}$ such that $f(A) = \mathbb{C}\backslash\{0\}$. (10 presentation points)