Assignment 3: Exam 1: Complex Numbers
Due Wednesday, February 8, 2023

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Problem 1 (Exercise 1.6.9 in BC) Show that if $|z| \geq 2$, then

$$|z^4 - 4z^2 + 3| \geq 3.$$

Determine conditions for equality. (10 presentation points)

Problem 2 (Exercise 1.6.10 in BC) Draw the set

$$\{z \in \mathbb{C} : \overline{z}^2 = z^2\}$$

in the complex plane. (10 presentation points)

Problem 3 (Example 1.5.3 in BC) Draw a picture illustrating the growth of a polynomial function as described by the assertion in Example 1.5.3 in BC. (10 presentation points)

Problem 4 (Exercise 1.9.1 in BC) Find the argument θ of

$$z = -\frac{1}{1+i}$$

satisfying the following conditions:

(a) $-\pi < \theta \leq \pi$.

(b) $0 \leq \theta < 2\pi$.

(10 presentation points)

Problem 5 (Exercise 1.9.4 in BC) Consider the equation

$$|e^{i\theta} - 1| = 2.$$

(a) Solve the equation geometrically.

(b) Solve the equation analytically.

(10 presentation points)

Problem 6 (Exercise 1.9.7 in BC) Use mathematical induction to show that if $z = re^{i\theta}$, then

(a) $z^n = r^n e^{in\theta}$ and

(b) $z^{-n} = (z^{-1})^n = (z^n)^{-1} = r^{-n}e^{-in\theta}$

for $n \in \mathbb{N}$. (10 presentation points)

Problem 7 (Exercise 1.11.1 in BC) Find the two complex square roots of i. (10 presentation points)

Problem 8 (Exercise 47 in my notes and Exercise 1.11.5 in BC) Let $\zeta = \sqrt{5}(1+i)$. Compute the following:

(a) ζ^3.

(b) The three complex cube roots of $10\sqrt{5}(-1+i)$.

(10 presentation points)

Problem 9 (Exercise 1.11.6 in BC) Given $n \in \mathbb{N}$, factor the polynomial $p(z) = z^{2n} - 1$ into a product of n quadratic polynomials with real coefficients. Hint(s): Let $1, \zeta_1, \zeta_2, \ldots, \zeta_{2n-1}$ be the complex roots of unity. Consider the first few cases. (10 presentation points)

Problem 10 (Exercise 1.12.1 in BC) For the following two subsets of $\mathbb{C}$, find the associated sets of interior points, exterior points, boundary points, accumulation points, closure and interior.

(a) $\{z \in \mathbb{C} : \operatorname{Im} z > 1\}$.

(b) $\{z \in \mathbb{C} : \operatorname{Re} z \geq 1\}$.

(10 presentation points)