Assignment 2: Complex Numbers
Due Wednesday, February 1, 2023

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Problem 1 (Exercises 1.31-34 from my notes and sections 1.4-6, Exercise 1.5.2,4,8 in BC) Show the following hold for complex numbers $z, w \in \mathbb{C}$:

(a) $\operatorname{Re}(z) \le |\operatorname{Re}(z)| \le |z|$.

(b) $\operatorname{Im}(z) \le |\operatorname{Im}(z)| \le |z|$.

(c) $|zw| = |z|\ |w|$.

(d) $|\overline{z}| = |z|$

(e) $|\operatorname{Re}(z)| + |\operatorname{Im}(z)| \le \sqrt{2}\ |z|$.

(10 presentation points)

Problem 2 Characterize cases of equality in the inequalities **(a)**, **(b)** and **(e)** in Problem 1 above. (10 presentation points)

Problem 3 (Exercise 1.37 from my notes) Consider the following conditions on a pair of complex numbers z and w:

A $z\overline{w} \geq 0$, i.e., there is some $\alpha \geq 0$ for which $z\overline{w} = \alpha$.

B One of the following three conditions holds:

- **(i)** $z = 0$.
- **(ii)** $w = 0$.
- **(iii)** There is some $a > 0$ for which $z = aw$.

C One of the following three conditions holds:

- **(i)** $z = 0$.
- **(ii)** $w = 0$.
- **(iii)** There is some $b > 0$ for which $w = bz$.

Show conditions **A**, **B**, and **C** are equivalent. (10 presentation points)

Problem 4 (Exercise 1.5.1 in BC) For each part draw the complex numbers z, w, $z + w$, $z - w$, and $w - z$ with the relevant parallelograms in the complex plane.

(a) $z = 2i$, $w = 2/3 - i$.

(b) $z = -sqrt3 + i$, $w = \sqrt{3}$.

(c) $z = -3 + i$, $w = 1 + 4i$.

(d) $z = x + yi$, $w = x - iy$.

(8 presentation points)

Problem 5 (Exercises 1.5.5,6 and 1.6.14 in BC) For each part draw the points z satisfying the relation in the complex plane:

(a) $|z - 1 + i| = 1$.

(b) $|z + i| \leq 3$.

(c) $|z - 4i| \geq 4$.

(d) $|z - 1| = |z + i|$.

(e) $z^2 + \overline{z}^2 = 2$.

(10 presentation points)

Problem 6 For each part in Problem 5 above, express the set of points under consideration properly in set notation. (10 presentation points)

Problem 7 (Exercise 1.30 from my notes) Let $z_0 \in \mathbb{C}\backslash\{0\}$ be fixed. Draw the set of points in the complex plane determined by the relations

$$\{z \in \mathbb{C} : |z + z_0| \leq |z| + |z_0|\} \qquad \text{and} \qquad \{z \in \mathbb{C} : |z + z_0| = |z| + |z_0|\}.$$

(10 presentation points)

Problem 8 (Exercise 1.38 in my notes and Example 1.5.3 in BC) Show the following:

(a) If $z \in \mathbb{C}\backslash\{0\}$, then

$$\left|\frac{1}{z}\right| = \frac{1}{|z|}.$$

(b) If $P : \mathbb{C} \to \mathbb{C}$ is a polynomial of order $n \geq 1$ with complex coefficients, then there is some $R > 0$ for which P has no zero in $\mathbb{C}\backslash B_R(0) = \{z \in \mathbb{C} : |z| \geq R\}$.

(10 presentation points)

Problem 9 (Exercise 1.40 in my notes and Exercises 1.6.1-6,13 in BC) Verify the following for complex numbers $z, w \in \mathbb{C}$:

(a) $\overline{zw} = \overline{z}\,\overline{w}$.

(b) $\overline{z/w} = \overline{z}/\overline{w}$ if $w \neq 0$.

(c) $|(2\overline{z} + 5)(\sqrt{2} - i)| = \sqrt{3}|2z + 5|$.

(d) $|z - w|^2 = |z|^2 - 2\,\mathrm{Re}(z\overline{w}) + |w|^2$.

(10 presentation points)

Problem 10 Is the function $g : \mathbb{C} \to \mathbb{C}$ by $g(z) = \overline{z}$ linear? Note: State carefully the possible interpretations of the question, i.e., in what way(s) $\mathbb{C}$ may be considered a vector space. (10 presentation points)

Problem 11 (Exercise 1.5.7 in BC and Exercise 1.39 in my notes) Consider the polynomial

$$P(z) = \sum_{j=0}^{n} a_j z^j$$

with $n \geq 1$ and $a_n \neq 0$. Show that for $R > 0$ large enough

$$\frac{7}{8}|a_n|R^n < |P(z)| < (1.01)|a_n||z|^n \qquad \text{for} \qquad |z| > R.$$

(10 presentation points)