Assignment 1: Complex Numbers
Due Wednesday, January 25, 2023

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Problem 1 (Exercise 1.2 from my notes and Exercise 1.2.6(a) in BC) Show addition in $\mathbb{C}$ is commutative. (5 presentation points)

Problem 2 (Exercise 1.3 from my notes)

(a) Define what it means for a set G with an operation

$$(a, b) \mapsto a * b$$

to be an algebraic **group**.

(b) Show that the complex numbers $\mathbb{C}$ form a group under addition.

(10 presentation points)

Problem 3 (Exercise 1.4 from my notes and Exercise 1.2.8 in BC) Show the identity element in any group is unique. (5 presentation points)

Problem 4 (Exercises 1.2.1-4 in BC) Express the following algebraic expressions involving complex numbers in the form $a + bi$ where $a, b \in \mathbb{R}$:

(a) $(\sqrt{2} - i) - i(1 - \sqrt{2}\, i)$.

(b) $(2 - 3i)(-2 + i)$.

(c) $[(3 + i)(3 - i)][1/5 + (1/10)i]$.

(d) $i(x + yi)$ where $i = 0 + 1i$ and $x, y \in \mathbb{R}$.

(e) $(1+x+yi)^2$ where $x, y \in \mathbb{R}$.

(f) $1+2(x+iy)+(x+yi)^2$ where $x, y \in \mathbb{R}$.

(g) $(1 \pm i)^2 - 2(1 \pm i) + 2$.

(h) $(i)(i)$.

(8 presentation points)

Problem 5 (Exercise 1.6 from my notes and Exercise 1.2.5 in BC) Show multiplication in $\mathbb{C}$ is associative and commutative. (10 presentation points)

Problem 6 (Exercises 1.7-10 from my notes and Exercise 1.2.8(b) in BC)

(a) Show $\mathbb{C}$ is **not** a group under multiplication.

(b) Show the multiplicative identity in $\mathbb{C}$ is unique.

(c) Given $x+yi \in \mathbb{C}\backslash\{0\}$ with $x, y \in \mathbb{R}$, express

$$\frac{1}{x+yi}$$

in the form $a+bi$ with $a, b \in \mathbb{R}$.

(10 presentation points)

Problem 7 (Exercise 1.16 from my notes and Exercise 1.2.1 in BC) Take each algebraic expression in Problem 4 above and translate it into the notation using ordered pairs of real numbers to denote elements of $\mathbb{C}$ introduced by Brown and Churchill; simplify the expression using the complex product on $\mathbb{R}^2$ and then write the result in both forms. (8 presentation points)

Problem 8 Consider the (real) vector space isomorphism $\gamma : \mathbb{R}^2 \to \mathbb{C}$ given by

$$\gamma(x, y) = x + yi.$$

(a) The function $I : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$I(x, y) = \gamma^{-1}(\gamma(1, 0)\gamma(x, y))$$

is a linear function. Identify this (well-known) linear function on the real plane and express the value of I in terms of matrix multiplication (as used for example in linear algebra).

(b) The function $J : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$J(x, y) = \gamma^{-1}(\gamma(0, 1)\gamma(x, y))$$

is a linear function. Identify this (well-known) linear function on the real plane and express the value of J in terms of matrix multiplication.

(10 presentation points)

Problem 9 (Exercise 1.2.11 in BC) Fill in the blanks to learn a new way to solve the quadratic equation $x^2 + x + 1 = 0$.

We look for real numbers a and b such that

$$(a + bi)^2 \ + \ a + bi \ + \ 1 = 0. \tag{1}$$

Expanding the square and writing the left side of (1) in the form $A+Bi$ with $A, B \in \mathbb{R}$, we have

$$\underline{\hspace{12cm}} = 0.$$

This complex equation is equivalent to the system

$$\begin{cases} \underline{\hspace{2cm}} \ (2a + 1) & = \ 0 \\ \underline{\hspace{7cm}} & = \ 0. \end{cases} \tag{2}$$

If $b = 0$, then $\underline{\hspace{7cm}} = 0$, but

$$\underline{\hspace{7cm}} = \left(a + \frac{1}{2}\right)^2 - \frac{1}{4} + 1 \ \underline{\hspace{2cm}} \ 0,$$

so it cannot be the case that $b = 0$.

Therefore, $\underline{\hspace{3.5cm}} = 0$ and

$$a = \underline{\hspace{2cm}}.$$

Thus, the second equation in the system (2) tells us

$$\underline{\hspace{4cm}} = 0 \qquad \text{and} \qquad b^2 = \underline{\hspace{1cm}}\ .$$

Consequently,

$$b = \underline{\hspace{1.5cm}} \qquad \text{and} \qquad a + bi = \underline{\hspace{3.5cm}}\ .$$

(10 presentation points)

Problem 10 (Exercise 1.3.4 in BC) Use mathematical induction to prove the following

If $z_1, z_2, \ldots, z_n \in \mathbb{C}$ and $z_1 z_2 \cdots z_n = 0$, then one of the n factors is zero.

(10 presentation points)

Problem 11 (Exercise 1.3.8 in BC) Use induction to prove the binomial theorem. (10 presentation points)

Problem 12 (Exercise 1.3.5 in BC) Let $z = a + bi \in \mathbb{C}$ and $w = x + iy \in \mathbb{C}$ with $a = \mathrm{Re}(z)$, $b = \mathrm{Im}(z)$, $x = \mathrm{Re}(w)$, and $y = \mathrm{Im}(w)$. Show the following:

(a) $z\overline{z} = |z|^2$.

(b) If $w \neq 0$, then

$$\frac{z}{w} = \frac{z\overline{w}}{|w|^2} = \frac{ax + by}{x^2 + y^2} - i\,\frac{ay - bx}{x^2 + y^2}.$$

(10 presentation points)