Associativity of the Symmetric Difference of Sets

A Proof from the Book

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February 7, 2020

Introduction

Paul Erdős is credited with the idea of proofs "from the book." I do not know if Erdős, who referred to "**The Book** in which God keeps the most elegant proof of each mathematical theorem," expounded any of the details of his conception of "the book." He was probably wise enough not to do so, especially given his view of the book as belonging to God. I have, first of all, a much more humanistic view of "the book." Still, I do associate with it a certain inaccessibility. It is a book not everyone gets to see. Perhaps the author does not wish for everyone to see it. A proof "from the book" is one to which the author has dovoted time and effort. I see someone—perhaps not a young someone—with patient insight writing for a special audience, taking the time and care to present a proof with clarity and more broadly, as Erdős said, elegance. I imagine it being composed with pictures at least drawn with colored pencils on high quality paper—perhaps on vellum with water colors. It is not posted on the internet or, in my conception, adorned with typeset text and computer generated figures. Nevertheless, the presentation here is all of these things, but still, I try to capture something of clarity (and elegance).

And moreover, I have in mind to communicate it to someone special. Someone who writes a proof "from the book," should be aware that knowledge is not neutral. It can be dangerous, and clarity about ideas is used, more often than not, by evil people to do evil and make the world worse for at least some others, and perhaps consequently for everyone. The proof recorded here is not an extremely deep one with particularly mysterious insights; its potential to be used in an evil way may not be that great. Nevertheless, it does contain some spark of clarity, and it is my sincere hope that the reader can make himself a special enough person to avoid using whatever it offers for evil purposes. Remember, not many people are that special.

The result for which I offer a proof is simply the following:

> The **symmetric difference** defined on pairs of sets A and B by
>
> $$A\Delta B = (A \cup B)\backslash(A \cap B) \tag{1}$$
>
> is an **associative** operation. That is,
>
> $$(A\Delta B)\Delta C = A\Delta(B\Delta C). \tag{2}$$

It is relatively easy to see that $A\Delta B$ may be defined using the alternative, and sometimes quite useful, expression

$$A\Delta B = (A\backslash B) \cup (B\backslash A). \tag{3}$$

Both expressions (1) and (3) are illustrated on the left in Figure 1. If one uses either of these expressions to write out either side of (2) using only intersection, union, and relative complement in full detail, the expressions obtained are, on the one hand,

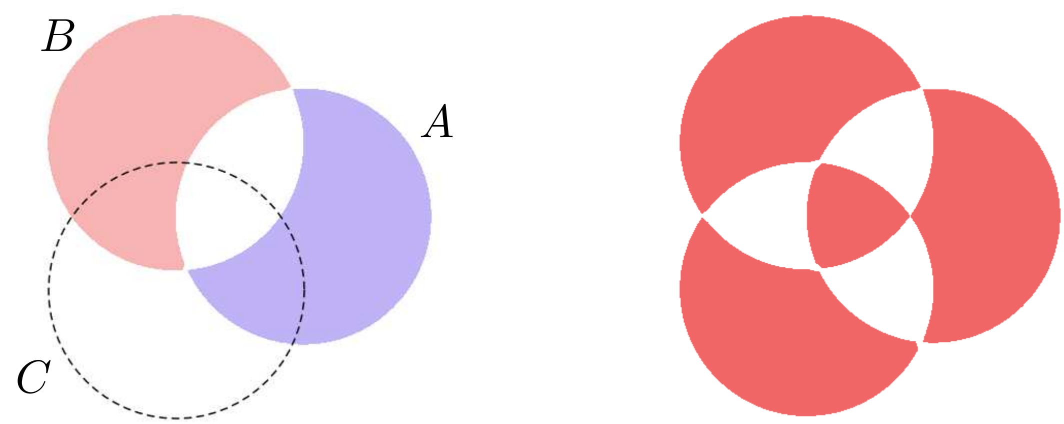

Figure 1: The symmetric difference of two sets $A\Delta B$ (left) and the symmetric difference of three sets $A\Delta B\Delta C$ (right).

complicated and, on the other hand, not at all obviously equal. Both, however, are illustrated on the right in Figure 1, and connecting each of the expressions $(A\Delta B)\Delta C$

and $A\Delta(B\Delta C)$ to that illustration comprises the content of our proof as explained in detail below.

It may be noted that in the second, alternative, expression for $A\Delta B$ the two sets into which the symmetric difference are decomposed are disjoint. We can use the "coproduct" symbol "$\coprod$" in place of a union symbol to emphasize the union of these sets is disjoint:

$$A\Delta B = (A\backslash B) \coprod (B\backslash A).$$

Generally, this notation is unfamiliar and distracting, but we will use it as an alternative in our proof when it illustrates something important.

Symmetric Difference as an Operation

It is perhaps worth noting that when one considers the symmetric difference Δ as an **operation** on two sets A and B, it is natural to assume there is a larger collection $\mathcal{C}$ of sets containing both A and B and the set $A\Delta B$, so that we may consider

$$\Delta : \mathcal{C} \times \mathcal{C} \to \mathcal{C}.$$

In this case, various other sets are involved, and some of these, C, $B\Delta C$ and $A\Delta B\Delta C$, in particular, should also be found in $\mathcal{C}$. One obvious way to proceed is to take $S = A \cup B \cup C$, or more generally

$$S = \bigcup_{E\in\mathcal{C}} E,$$

and let the collection $\mathcal{P}(S)$ of all subsets[1] of S be the collection under consideration. That is, $A, B, C \subset S$ and

$$\Delta : \mathcal{P}(S) \times \mathcal{P}(S) \to \mathcal{P}(S).$$

This provides a suitable context in which to speak of the symmetric difference as an operation.

The Proof

As suggested above, our strategy is to show the sets

[1]This is called the **power set** of S and is also denoted by $\mathcal{P}(S) = 2^S$.

$$(A\Delta B)\Delta C = [(A\Delta B)\backslash C] \cup [C\backslash (A\Delta B)] \tag{4}$$

and

$$A\Delta(B\Delta C) = [A\backslash (C\Delta B)] \cup [(B\Delta C)\backslash A] \tag{5}$$

are equal to the same set, namely the set represented on the right in Figure 1. Notice we have used the second expression (3) in both (4) and (5). We can sharpen this strategy in each case. The set on the right in Figure 1 is

$$[A\backslash (B \cup C)] \cup [B\backslash (A \cup C)] \cup [C\backslash (A \cup B)] \cup (A \cap B \cap C) \tag{6}$$

which we have recolored in Figure 2.

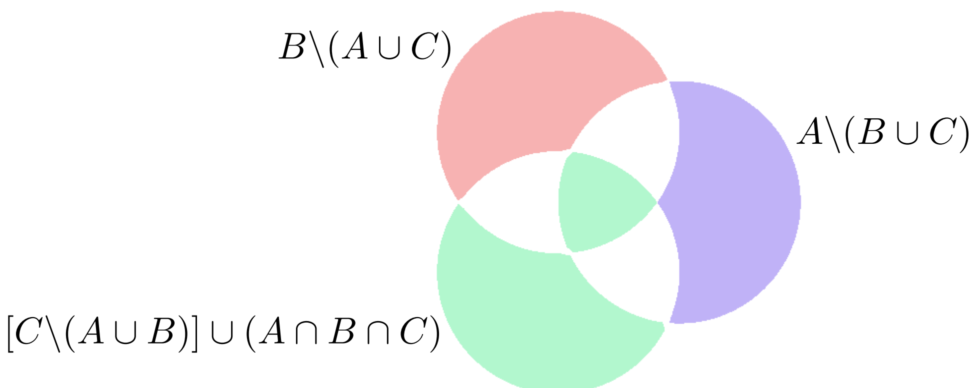

Figure 2: The symmetric difference of three sets with $A\backslash(B\cup C)$ (in blue), $B\backslash(A\cup C)$ (in red), and $[C\backslash(A \cup B)] \cup (A \cap B \cap C)$ (in green).

If we wish to emphasize that these sets whose union is the triple symmetric difference are disjoint, we can write:

$$[A\backslash(B \cup C)] \coprod [B\backslash(A \cup C)] \coprod [C\backslash(A \cup B)] \coprod (A \cap B \cap C)$$

With regard to (4) we propose to show

$$(A\Delta B)\backslash C = [A\backslash(B \cup C)] \cup [B\backslash(A \cup C)] \tag{7}$$

and

$$C\backslash(A\Delta B) = [C\backslash(A \cup B)] \cup (A \cap B \cap C). \tag{8}$$

In order to prove (7) and (8) from which it clearly follows that the expressions in (4) and (6) give the same set, we verify some simpler set identities.

Lemma 1 *For any sets C, D, and E, we have the following:*

(a) $(D \cup E) \backslash C = (D \backslash C) \cup (E \backslash C)$.

(b) $(D \backslash E) \backslash C = D \backslash (E \cup C)$.

(c) $C \backslash (D \backslash E) = (C \backslash D) \cup (C \cap D \cap E)$.

The identities (a) and (b) illustrate that while **relative complement** has a kind of right distributive law (a) with respect to unions, the operation is not symmetric. The identity (b) may be extended by De Morgan's law to the expression $(D \backslash E) \cap (D \backslash C)$, though we will not need this form. The identities (b) and (c) may be interpreted to illustrate that relative complement is not associative.
Proof of (a):

$$\begin{aligned} x \in (D \cup E) \backslash C \quad &\iff \quad (x \in D \text{ or } x \in E) \text{ and } x \notin C \\ &\iff \quad (x \in D \backslash C) \text{ or } (x \in E \backslash C) \\ &\iff \quad x \in (D \backslash C) \cup (E \backslash C) \qquad \Box \end{aligned}$$

Proof of (b): On the one hand, $x \in (D \backslash E) \backslash C$ implies $x \in D$ and $x \notin E$. Also, $x \in (D \backslash E) \backslash C$ implies $x \notin C$. Therefore,

$$x \in (D \backslash E) \backslash C \quad \Longrightarrow \quad x \in D \backslash (E \cup C).$$

On the other hand, $x \in D \backslash (E \cup C)$ implies $x \in D$ and $x \notin E$ and $x \notin C$. Thus,

$$x \in D \backslash (E \cup C) \quad \Longrightarrow \quad x \in (D \backslash E) \backslash C. \qquad \Box$$

Proof of (c): Note first that

$$x \notin D \backslash E \quad \iff \quad (x \notin D \text{ or } x \in D \cap E).$$

Therefore,

$$\begin{aligned} x \in C \backslash (D \backslash E) \quad &\iff \quad x \in C \text{ and } x \notin (D \backslash E) \\ &\iff \quad x \in C \text{ and } (x \notin D \text{ or } x \in D \cap E) \\ &\iff \quad (x \in C \backslash D) \text{ or } x \in C \cap D \cap E \\ &\iff \quad x \in (C \backslash D) \cup (C \cap D \cap E). \qquad \Box \end{aligned}$$

Proof of (7): We begin by expanding $A\Delta B$ using the second alternative expression giving a disjoint union.

$$\begin{aligned}(A\Delta B)\backslash C &= [(A\backslash B)\cup(B\backslash A)]\backslash C \\ &= [(A\backslash B)\backslash C]\cup[(B\backslash A)\backslash C] && \text{by (a)} \\ &= [A\backslash(B\cup C)]\cup[B\backslash(A\cup C)] && \text{by (b).} \qquad \square\end{aligned}$$

Proof of (8): This time we expand $A\Delta B$ using the first defining expression for the symmetric difference:

$$\begin{aligned}C\backslash(A\Delta B) &= C\backslash[(A\cup B)\backslash(A\cap B)] \\ &= [C\backslash(A\cup B)]\cup(A\cap B\cap C) && \text{by (c).} \qquad \square\end{aligned}$$

It remains to show the sets in (5) and (6) are the same. This follows from the decompositions

$$A\backslash(B\Delta C) = (A\cap B\cap C)\cup[A\backslash(B\cup C)] \tag{9}$$

and

$$(B\Delta C)\backslash A = [B\backslash(A\cup C)]\cup[C\backslash(A\cup B)] \tag{10}$$

which are very similar to (7) and (8). We leave the associated recoloring and verification of (9) and (10) to the reader.