Solution of Problem 14 from Assignment 2A
Analysis I Spring 2020

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April 23, 2020

1 Introduction

Here is the original statement of the problem:

Consider $u_n : \mathbb{R} \to \mathbb{R}$ by

$$u_n(x) = \begin{cases} -1/n^2, & x < 1/n \\ 1/n^2, & x \geq 1/n \end{cases} \qquad \text{for } n \in \mathbb{N}.$$

1. Plot (draw the graph of)

$$f_k(x) = \sum_{n=1}^{k} u_n(x)$$

   for $k = 1, 2, 3, 4$.

2. Does

$$f(x) = \sum_{n=1}^{\infty} u_n(x)$$

   make sense as a non-decreasing function? If so what is the set of discontinuities of f?

2 Preliminaries

A previous problem asserts that if u and v are non-decreasing, then

$$x_0 \text{ is a point of discontinuity for } u \implies x_0 \text{ is a point of discontinuity for } u + v$$

and

$$x_0 \text{ is a point of continuity for } u \text{ and } v \implies x_0 \text{ is a point of continuity for } u+v.$$

By induction these apply to finite sums of functions as follows: If $u_1, u_2, \ldots u_k$ are non-decreasing functions, then

$$x_0 \text{ is a point of discontinuity for } \textbf{any one of the functions } u_1, u_2, \ldots, u_k$$
$$\implies x_0 \text{ is a point of discontinuity for } f_k = \sum_{j=1}^{k} u_j$$

and

$$x_0 \text{ is a point of continuity for } \textbf{all of the functions } u_1, u_2, \ldots, u_k$$
$$\implies x_0 \text{ is a point of continuity for } f_k = \sum_{j=1}^{k} u_j.$$

3 Solution

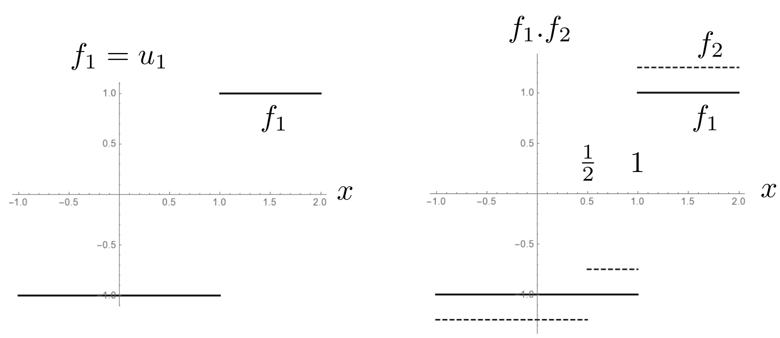

Figure 1: Plots of f_1 and f_2

Let $n \in \mathbb{N}$ be fixed. Notice that for $x \geq 1/n$ and $j \geq n$, we know $u_j(x) = 1/j^2$. Thus, for $k > n$

$$f_k(x) = \sum_{j=1}^{n-1} u_j(x) + \sum_{j=n}^{k} \frac{1}{j^2}. \tag{1}$$

In particular, $f_k(x) \leq f_{k+1}(x)$ for $k > n$. Therefore, either $\{f_k(x)\}_{k>n}$ is bounded above or not bounded above. We will show this sequence is bounded above:

$$\begin{aligned}
\sum_{j=n}^{k} \frac{1}{j^2} &= \frac{1}{n^2} + \sum_{j=n+1}^{k} \frac{1}{j^2} \\
&\leq \frac{1}{n^2} + \sum_{j=n+1}^{k} \frac{1}{(j-1)j} \\
&= \frac{1}{n^2} + \sum_{j=n+1}^{k} \left(\frac{1}{j-1} - \frac{1}{j}\right) \\
&= \frac{1}{n^2} + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots + \left(\frac{1}{k-1} - \frac{1}{k}\right) \\
&= \frac{1}{n^2} + \frac{1}{n} + \left(-\frac{1}{n+1} + \frac{1}{n+1}\right) + \left(-\frac{1}{n+2} + \frac{1}{n+2}\right) + \cdots + \left(-\frac{1}{k-1} + \frac{1}{k-1}\right) - \frac{1}{k} \\
&= \frac{1}{n^2} + \frac{1}{n} - \frac{1}{k} \\
&\leq \frac{n+1}{n^2}.
\end{aligned}$$

Since n is a fixed constant, so is $\sum_{j=1}^{n-1} u_j(x)$, and for $x \geq 1/n$ and $k > n$,

$$f_k(x) \leq \sum_{j=1}^{n-1} u_j(x) + \frac{n+1}{n^2} < \infty.$$

Thus, for $x \geq 1/n$ the sum

$$f(x) = \sum_{j=1}^{\infty} u_j(x) \quad \text{is a finite number.}$$

Since $n \in \mathbb{N}$ was arbitrary, $f(x)$ is given by the same formula for $x > 0$.

On the other hand, for $x < 0$, we know $u_j(x) = -1/j^2$ for all j. Thus, $f_{k+1}(x) < f_k(x)$ and $\{f_k(x)\}_{k>n}$ is either bounded below or not bounded above. We will show this sequence is bounded below:

$$\begin{aligned}\sum_{j=1}^{k} u_j(x) &= -1 - \sum_{j=2}^{k} \frac{1}{j^2} \\ &\geq -1 - \sum_{j=2}^{k} \frac{1}{(j-1)j} \\ &= -1 - \sum_{j=2}^{k} \left(\frac{1}{j-1} - \frac{1}{j} \right) \\ &= -1 - \left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{3}\right) - \cdots - \left(\frac{1}{k-1} - \frac{1}{k}\right) \\ &= -2 + \left(\frac{1}{2} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{k-1} - \frac{1}{k-1}\right) + \frac{1}{k} \\ &= -2 + \frac{1}{k} \\ &\geq -2.\end{aligned}$$

This means that for $x \leq 0$, not only do we know

$$f(x) = \sum_{j=1}^{\infty} u_j(x) = -\sum_{j=1}^{\infty} \frac{1}{j^2} \quad \text{is a finite number,}$$

but we also know f takes only this constant value on $(-\infty, 0]$. In particular, f is continuous at each $x < 0$. Let's write the negative real number[1] $f(0)$ as $f(0) = -\pi/6$. More generally, for each $n \in \mathbb{N}$ and $x \geq 1/n$ the value of $f(x)$ may be expressed as

$$f(x) = \sum_{j=1}^{n-1} u_j(x) + \sum_{j=n}^{\infty} \frac{1}{j^2} = \sum_{j=1}^{n-1} u_j(x) + \frac{\pi(n)}{6} \tag{2}$$

where $\pi(n)$ is the unique well-defined positive number given by

$$\pi(n) = 6 \sum_{j=n}^{\infty} \frac{1}{j^2} \leq \pi(1) = \pi$$

[1]To prove $\sum_{j=1}^{\infty} 1/j^2 = \pi/6$ is called **the Basel Problem** after the city of Basel in Switzerland where Euler and the Bernoulli's were from. For our purposes, we can just introduce π here as a symbol to denote $6\sum_{j=1}^{\infty} 1/j^2 = f(1)$ which we have shown is a well-defined finite positive real number.

where equality holds only for $n = 1$.

Notice that for $x > 0$, we can take $n \in \mathbb{N}$ with $1/n \leq x$ so that by (1)

$$f(x) \geq f_k(x) = \sum_{j=1}^{n-1} u_j(x) + \sum_{j=n}^{k} \frac{1}{j^2} \geq \sum_{j=1}^{n-1} u_j(x) \geq -\sum_{j=1}^{n-1} \frac{1}{j^2} > -\frac{\pi}{6} = f(0).$$

Also, if $0 < x_1 < x_2$, then we may take $n \in \mathbb{N}$ with $1/n \leq x_1$ so that

$$f(x_1) = \sum_{j=1}^{n-1} u_j(x_1) + \frac{\pi(n)}{6} \leq \sum_{j=1}^{n-1} u_j(x_2) + \frac{\pi(n)}{6} = f(x_2)$$

since $\sum_{j=1}^{n-1} u_j$ is a finite sum of non-decreasing functions. We have now verified that $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} -\pi/6, & x \leq 0 \\ \sum_{j=1}^{\infty} u_j(x), & x > 0 \end{cases}$$

is a well-defined non-decreasing function which is continuous at each point x with $x < 0$.

We claim next that f is continuous at $x = 0$. Notice that since $\pi(n) > 0$, the expression (2) gives us an estimate for $f(x)$ for each $n \geq 1/x$, namely

$$f(x) \leq f(1/n) < \sum_{j=1}^{n-1} u_j(1/n) = -\sum_{j=1}^{n-1} \frac{1}{j^2}.$$

Let $\epsilon > 0$. Taking n large enough so that

$$-\sum_{j=1}^{n-1} \frac{1}{j^2} < -\frac{\pi}{6} + \epsilon,$$

we can take $\delta = 1/n > 0$. Then for $|x| < \delta$,

$$|f(x) - f(0)| \leq f(|x|) + \frac{\pi}{6} < \epsilon.$$

Thus, f is continuous at $x = 0$. We claim, finally, that f is discontinuous precisely on the set

$$\Gamma = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

In particular, $f \in C^0(\mathbb{R}\backslash\Gamma)$. The key idea to see this is already evident in Figure 1, but we will illustrate it again in the case $k = 4$ in accord with the instructions of the problem:

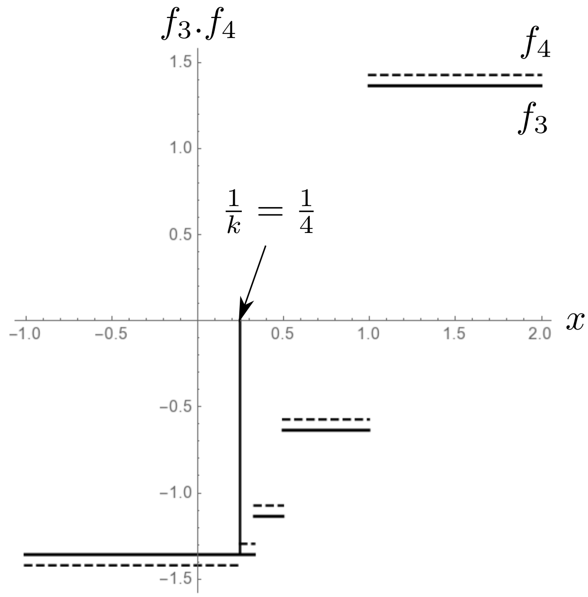

Figure 2: Plots of f_3 (solid) and f_4 (dashed). The basic/important idea here is that when you add u_k to f_{k-1}, in this case u_4 to f_3, every value $f_k(x)$ for $x \geq 1/k$ is precisely equal to $f_{k-1}(x)$ plus a constant. In fact, for $x \geq 1/k$ we have $f_k(x) = f_{k-1}(x)+1/k^2$. In the figure it may be observed that for $x \geq 1/4$, we have $f_4(x)+1/16$. Consequently, all points of continuity for f_{k-1} are points of continuity for f_k; all points of discontinuity for f_{k-1} are points of continuity for f_k. This idea carries over to the infinite sum because $f(x)$ for $x > 1/k$ is also presisely equal to $f_{k-1}(x)$ plus a constant.

For any $x_0 > 0$, let $n \in \mathbb{N}\backslash\{1\}$ with $1/n < x_0$. Then according to (2) for any $x \geq 1/n$,

$$f(x) = \sum_{j=1}^{n-1} u_j(x) + \frac{\pi(n)}{6}. \tag{3}$$

Notice that the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \sum_{j=1}^{n-1} u_j(x) + \frac{\pi(n)}{6}$$

is a finite sum of non-decreasing functions with discontinuities precisely in the set $\Gamma_n = \{1, 1/2, \ldots, 1/n - 1\}$. Note well, that the functions f and g are not equal for all $x \in \mathbb{R}$, but they are equal for $x \geq 1/n$. In particular, if $x_0 \in \Gamma_n$, then

$$\sup\{f(x) : x < x_0\} = \sup\{g(x) : x < x_0\} < \inf\{g(x) : x > x_0\} = \inf\{f(x) : x > x_0\},$$

so f has a discontinuity at x_0. Similarly, if $x_0 \notin \Gamma_n$, then for any $\epsilon > 0$, we can take $\delta_1 > 0$ with $x_0 - \delta_1 > 1/n$ so that for every $x \in \mathbb{R}$ with $|x - x_0| < \delta_1$, we have[2]

$$x > x_0 - \delta_1 > 1/n,$$

and (3) holds. By continuity of the function g, we can take $\delta > 0$ with $\delta < \delta_1$ such that

$$|x - x_0| < \delta \qquad \Longrightarrow \qquad |g(x) - g(x_0)| < \epsilon.$$

Equivalently, we could say

$$\sup\{g(x) : x < x_0\} = \inf\{g(x) : x > x_0\}.$$

Either way, for x with $|x - x_0| < \delta$, we know $f(x) = g(x)$, so

$$|x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \epsilon$$

and

$$\sup\{f(x) : x < x_0\} = \inf\{f(x) : x > x_0\}.$$

That is, f is continuous at x_0. $\square$

[2]If $x \leq x_0 - \delta_1$, then $x_0 - x \geq \delta_1 > 0$, so $|x - x_0| = x_0 - x \geq \delta_1$ which is a contradiction.