Math 4317, Self Assessment 3

Incompleteness of $\mathbb{Q}$

Let

$$A = \{x \in \mathbb{Q} : x^2 \leq 2\}$$

where $\mathbb{Q}$ denotes the rational numbers.

1. Show that if $b \in P = \{x \in \mathbb{Q} : x > 0\}$ is positive and $b^2 > 2$ then b is **not** the least upper bound of A, i.e., find a rational number c which is an upper bound for A with $c < b$. Hint: You can use the fact that if $c \in P$ and $c^2 \geq 2$, then c is an upper bound for A. (This was a problem on Self Assessment 2.)

Solution:

Pengfei Cheng gave the following elegant solution for this first question:

Since $b^2 > 2$, we know $b^2 - 2 > 0$, and we can find some $n \in \mathbb{N}$ such that

$$b^2 - 2 > \frac{1}{n}.$$

Then the rational number

$$c = b - \frac{1}{2bn}$$

satisfies

$$c^2 = \left(b - \frac{1}{2bn}\right)^2 = b^2 - \frac{1}{n} + \frac{1}{4b^2n^2} \geq b^2 - \frac{1}{n} > 2.$$

I would like to know how many of you can give a **different solution** along the following lines:

Let $c = b - 1/n$ where $n \in \mathbb{N}$ will be chosen later.

Can you show this c satisfies $c^2 > 2$ when n is large enough?

Also, there is an error in Pengfei's proof above. For extra credit: Find the error.