Construction of $\mathbb{R}$

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1 Making Things Strange

We have constructed the natural numbers, the integers, and the rational numbers. Now we will discuss the construction of the **real numbers** from the rationals. The last two constructions involved equivalence classes in a Cartesian product. Before we discuss the next extension it may be amusing to look back at the objects considered up to this point. We started, in some sense, with the number zero, which we took to be the empty set:

$$0 = \phi.$$

We also defined $1 = \{\phi\}$ and the other natural numbers. In that strange but simple world of sets, we knew $0 < 1$ because $\phi \subset \{\phi\}$ but $\{\phi\} \not\subset \phi$. After we constructed the integers, zero took the form

$$0 = [(0, 0)] = [(\phi, \phi)],$$

and we also had

$$1 = [(\{\phi\}, \phi)].$$

The brackets indicate equivalence classes, so these expressions carry with them some additional structure and could be made, in a certain sense, more explicit:

$$0 = \{(m, n) : m, n \in \mathbb{N}_0 \text{ with } m = n\}$$

It will be noted that to write zero in this way, we need to assume the set structure of all the natural numbers n and m. It has already become too cumbersome to write out the definition of the equivalence class in full detail, from scratch, going all the way back to the inductive definition of a natural number n in terms of $0 = \phi$. But we can still hold that structure pretty well in our minds when we see $m = n$ for $m, n \in \mathbb{N}_0$; we know what it means.

The next step up to the rational numbers involved another, different equivalence relation. Since we are about to use the two equivalence relations at the same time, we will do well to introduce a second notation for the equivalence classes with respect to the second equivalence relation. Let me suggest the use of "double square brackets" in the form $[\![\ ]\!]$ so that a rational number looks like

$$[\![(p, q)]\!]$$

where $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^*$. Then zero becomes

$$0 = [\![([(\phi, \phi)], [(\{\phi\}, \phi)])]\!].$$

Behold zero as a rational!

Exercise 1 *Write* 1, 2, *and* 1/2 *at this level of absurdity, that is, using only the set* ϕ *and the appropriate parentheses for sets and ordered pairs and the appropriate brackets for equivalence classes.*

Now, we are going to add onto this a final layer of complication, so you can see our minds will need to be very full of structure when we think of a real number. And this last extension is, in some sense, rather more complicated than the last two. In another way, it's somewhat similar, so let's ponder some of the similarities and difference before we actually define

$$\gamma : \mathbb{Q} \to \mathbb{R}.$$

One similarity is that the construction involves two steps and the second one is based on forming equivalence classes using an equivalence relation. So that's familiar. The set in which we find equivalence classes, however, is a bit more complicated than a Cartesian product. The equivalence relation is entirely different being, for the first time, fundamentally a matter of analysis rather than algebra.[1] Let us abstractly denote the two step procedure in this case by

$$\mathbb{Q} \to \Gamma \to \Gamma/\sim\, = \mathbb{R}.$$

[1] The first equivalence relation was used to add additive inverses to get a group structure. The second equivalence relation was used to add multiplicative inverses to get a field structure. This one will be used to add limits to fill in the holes in the rational numbers.

The set Γ is a subset of a Cartesian product with factors $\mathbb{Q}$, but instead of just two, there are infinitely many:

$$\Gamma \subset \prod_{n=1}^{\infty} \mathbb{Q}.$$

The elements of the Cartesian products $\mathbb{N}_0 \times \mathbb{N}_0$, $\mathbb{Z} \times \mathbb{Z}^*$, and $\mathbb{Q} \times \mathbb{Q}$ are ordered pairs or "2-tuples." Similarly, elements of this infinite Cartesian product may be thought of $\mathbb{N}$-tuples or functions in $\mathbb{Q}^{\mathbb{N}}$. These are **sequences**, and we shall denote a sequence in $\prod_{n=1}^{\infty} \mathbb{Q}$ by

$$\{a_n\}_{n=1}^{\infty}.$$

Notice we're using the curly parentheses that we also use for sets. There is a similarity between the set

$$\{a_1, a_2, a_3, \ldots\} \qquad \text{and} \qquad \{a_n\}_{n=1}^{\infty}.$$

They contain the same values of a $\mathbb{Q}$ valued function on $\mathbb{N}$. The sequence, however, has an order while the set does not. At least that is one difference. Another, is that the "elements" in the sequence carry with them tags. Thus, one may have $a_1 = a_5$ in a sequence, but a_1 and a_5 are different in the sequence. (These elements, being equal, would be the same in the set of values.) Thus, a sequence can be thought of as a countably infinite disjoint union of singletons $\{a_n\}$. I've probably said too much about this topic of sequences and sequence notation, but I will mention also that some authors do use a different notation for sequences, namely

$$(a_n) \qquad \text{or} \qquad (a_n)_{n=1}^{\infty}.$$

Either way, there is some duplication of parentheses.

Returning to our main subject and having pointed out that sequences are like ordered pairs in the sense of being $\mathbb{N}$-tuples and that we plan to define an equivalence relation on sequences, or some subset of the sequences with rational values, I think we have exhausted the similarities between the construction of the real numbers and our previous constructions. Now it is, more or less, time to give the construction. Before we do that, however, it is customary to say, abstractly, what kind of object we are going to construct and say something, abstractly, about it before we construct it. This is relatively short and to the point, so I will do it in the next section. If you want to jump directly to the explicit construction of $\mathbb{R}$ in the spirit our our previous constructions, skip down to the third section.

2 The Abstract Real Numbers

Abstractly the real numbers $\mathbb{R}$ are a **complete ordered field**. The condition of completeness can be formulated in various ways. For us, I will simply recall that in this context we have the following:

1. The ordering of our field comes from the identification of a **set of positives**.
2. Such an ordering is a total ordering.
3. Using the ordering, it is easy to define **upper and lower bounds** and **least upper and greatest lower bounds**.

Completeness can then be formulated as follows:

> A complete ordered field Φ is an ordered field which satisfies the **least upper bound property**, that is, whenever $A \subset \Phi$ is a nonempty set which is bounded above, there exists a least upper bound for A in Φ.

We have noted previously that $\mathbb{Q}$ is not complete. Here are the main abstract assertions:

Theorem 1 *There exists a complete ordered field.*

Theorem 2 *If F and Φ are two complete ordered fields, then there exists an (order preserving) field isomorphism $\psi : F \to \Phi$. Thus, up to field isomorphism, there is a* **unique** *complete ordered field.*

We could go on to say how there is a copy of $\mathbb{Q}$ in every complete ordered field (and hence a copy of $\mathbb{Z}$ and $\mathbb{N}$ as well), but we will save such observations for the concrete example, namely $\mathbb{R}$, we now construct.

3 The REAL Real Numbers

3.1 The set Γ

Let $\mathbb{Q}^N$ be the set of all sequences of rational numbers. A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be **Cauchy** if the following condition holds:

> For any $\epsilon > 0$ (in $\mathbb{Q}$) there is some $N \in \mathbb{N}$ such that
>
> $$m, n > N \text{ in } \mathbb{N} \qquad \Longrightarrow \qquad |a_m - a_n| < \epsilon.$$

The set Γ mentioned above is the set of all Cauchy sequences of rational numbers.

3.2 The equivalence relation

Two Cauchy sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are said to be **equivalent**, written (as usual)

$$\{a_n\}_{n=1}^{\infty} \sim \{b_n\}_{n=1}^{\infty},$$

if the following condition holds:

For any $\epsilon > 0$ (in $\mathbb{Q}$) there is some $N \in \mathbb{N}$ such that

$$m, n > N \text{ (in } \mathbb{N}) \qquad \Longrightarrow \qquad |a_m - b_n| < \epsilon.$$

3.3 The field $\mathbb{R}$

$$\mathbb{R} = \Gamma / \sim = \{[\{a_n\}_{n=1}^{\infty}] : \{a_n\}_{n=1}^{\infty} \in \Gamma\}.$$

Additional complications arise at this point. You may recall that we were able to state immediately that $\mathbb{Q}$ was a field after we defined the set $\mathbb{Q}$ in terms of the appropriate equivalence relation. In this case, it is initially difficult to identify the multiplicative inverses of elements in $\mathbb{R}^*$, so we proceed somewhat more slowly.

Theorem 3 *$\mathbb{R} = \Gamma / \sim$ is an ordered* **ring** *under the componentwise operations*

$$[\{a_n\}_{n=1}^{\infty}] + [\{b_n\}_{n=1}^{\infty}] = [\{a_n + b_n\}_{n=1}^{\infty}] \qquad \text{and} \qquad [\{a_n\}_{n=1}^{\infty}][\{b_n\}_{n=1}^{\infty}] = [\{a_n b_n\}_{n=1}^{\infty}]$$

with additive and multiplicative identities

$$[\{0\}_{n=1}^{\infty}] \qquad \text{and} \qquad [\{1\}_{n=1}^{\infty}]$$

respectively. The additive inverse of $[\{a_n\}_{n=1}^{\infty}]$ is given by

$$[\{-a_n\}_{n=1}^{\infty}].$$

A real number $[\{a_n\}_{n=1}^{\infty}]$ is said to be **positive** *if every Cauchy sequence of rationals*

$$\{\alpha_n\}_{n=1}^{\infty} \in [\{a_n\}_{n=1}^{\infty}]$$

has the following property:

There exist $M, N \in \mathbb{N}$ such that

$$n > N \qquad \Longrightarrow \qquad \alpha_n > \frac{1}{M}.$$

The resulting collection of positive reals makes $\mathbb{R}$ a **totally ordered ring**.

The multiplicative inverse of $[\{a_n\}_{n=1}^{\infty}] \in \mathbb{R}\backslash\{0\}$ is obtained as follows:

Lemma 1 *If $[\{a_n\}_{n=1}^{\infty}] \in \mathbb{R}^*$, then for each Cauchy sequence of rationals*

$$\{\alpha_n\}_{n=1}^{\infty} \in [\{a_n\}_{n=1}^{\infty}]$$

there is some $N \in \mathbb{N}$ such that $\alpha_n \in \mathbb{Q}^*$ for $n > N$. Consequently, we may set

$$b_n = \begin{cases} 1 & n \leq N \\ 1/\alpha_n & n > N. \end{cases}$$

This defines a Cauchy sequence $\{b_n\}_{n=1}^{\infty}$ and

$$[\{b_n\}_{n=1}^{\infty}][\{a_n\}_{n=1}^{\infty}] = [\{a_n\}_{n=1}^{\infty}][\{b_n\}_{n=1}^{\infty}] = [\{1\}_{n=1}^{\infty}].$$

With this result, the theorem above can be strengthened:

Theorem 4 $\mathbb{R} = \Gamma/\sim$ *is a* **complete** *totally ordered* **field** *with multiplicative inverses given in the preceding lemma.*

The completeness in this result requires some work. Given a nonempty collection A of bounded (equivalence classes of) sequences for which

$$[\{a_n\}_{n=1}^{\infty}] \leq [\{b_n\}_{n=1}^{\infty}] \qquad \text{for every} \qquad [\{a_n\}_{n=1}^{\infty}] \in A,$$

one must construct a Cauchy sequence of rationals whose equivalence class constitutes a least upper bound. One way to obtain such a sequence is to use the following observation:

Lemma 2 *If*

1. $B \subset \mathbb{Q}$,
2. $b \in \mathbb{Q}$,
3. $n \in \mathbb{N}$ *and*
4. $$B_n = \left\{ b - \frac{k}{n} : k \in \mathbb{N}_0 \right\} \cap B \text{ is nonempty and bounded below,}$$

then $\min B_n = b - k_n/n \in \mathbb{Q}$ *exists.*

Using the axiom of choice, we take a specific representative sequence of rationals

$$\{a_n^\alpha\}_{n=1}^\infty \in \alpha = [\{a_n\}_{n=1}^\infty]$$

from each equivalence class $\alpha \in A$.

Now we proceed to construct a sequence $\{c_k\}_{k=1}^\infty \in \mathbb{Q}^N$ inductively starting with $k = 1$. We first set

$$B_1 = \{b_1 - k : k \in \mathbb{N}_0\} \cap \{r \in \mathbb{Q} : a_n^\alpha \leq r \text{ for all } n \in \mathbb{N} \text{ and } \alpha \in A\}.$$

Then we define

$$c_1 = \begin{cases} b_1 & \text{if } b_1 < a_n^\alpha \text{ for some } n \in \mathbb{N} \text{ and } \alpha \in A \\ \min B_1 & \text{if } a_n^\alpha \leq b_1 \text{ for all } n \in \mathbb{N} \text{ and } \alpha \in A. \end{cases}$$

Notice that in the second case we have $b_1 \in B_1$ and B_1 is bounded below by any particular a_n^α, so B_1 is nonempty and bounded below and the lemma applies.

Having defined $B_1, \ldots, B_\ell$ and $c_1, \ldots, c_\ell$, we set

$$B_{\ell+1} = \left\{ b_{\ell+1} - \frac{k}{\ell+1} : k \in \mathbb{N}_0 \right\} \cap \{r \in \mathbb{Q} : a_n^\alpha \leq r \text{ for all } \alpha \in A \text{ and } n > \ell \text{ in } \mathbb{N}\}.$$

Then we define

$$c_{\ell+1} = \begin{cases} b_{\ell+1} & \text{if } b_{\ell+1} < a_n^\alpha \text{ for some } \alpha \in A \text{ and some } n \geq \ell+1 \text{ (in } \mathbb{N}) \\ \min B_1 & \text{if } a_n^\alpha \leq b_{\ell+1} \text{ for all } \alpha \in A \text{ and } n \geq \ell+1 \text{ (in } \mathbb{N}). \end{cases}$$

Again in the second case we have $b_{\ell+1} \in B_{\ell+1}$ and $B_{\ell+1}$ is bounded below by any particular a_n^α with $n \geq \ell+1$, so $B_{\ell+1}$ is nonempty and bounded below and the lemma applies.

Exercise 2 *Show the sequence* $\{c_k\}_{k=1}^\infty$ *is a Cauchy sequence in* $\mathbb{Q}$ *which is a least upper bound for* A. *More properly, the equivalence class* $[\{c_k\}_{k=1}^\infty]$ *of the sequence* $\{c_k\}_{k=1}^\infty$ *is a least upper bound for* A.

Theorem 5 $\gamma : \mathbb{Q} \to \mathbb{R}$ *where*

$$\gamma(r) = [\{r\}_{n=1}^\infty] \text{ is the equivalence class of the constant sequence,}$$

is an order preserving injection which is also a field homomorphism.[2]

[2] A field homomorphism is just a ring homomorphism of fields.

Evidently, the injection γ provides us with "copies" of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ in $\mathbb{R}$. It is worth noting that whenever you have an ordered field, then you can inject the integers (and the rationals) into it starting with a function γ satisfying $\gamma(0) = 0$ and $\gamma(1) = 1$ and working up from there under the assumption that γ is a field homomorphism.

Exercise 3 *Write $0 \in \mathbb{R}$ in terms of ϕ and three layers of equivalence classes according to our construction of $\mathbb{R}$ from "nothing."*

Finally, we note that we can consider Cauchy sequences of real numbers. Yes, that would be Cauchy sequences of equivalence classes of Cauchy sequences (of equivalence classes of pairs of equivalence classes of pairs of equivalence classes of natural numbers).

Theorem 6 *Every Cauchy sequence of real numbers converges to a unique real number.*

This result is sometimes called the **Cauchy completeness theorem**, and the condition that every Cauchy sequence converges (in a metric space) is called **metric completeness**. The fact that $\mathbb{R}$ is metrically complete is weaker than the fact that it is complete according to our condition concerning the existence of least upper bounds, which is called **Dedekind completeness**. In an ordered field F you have a copy of $\mathbb{N}$, and metric completeness together with the **Archimedean property** (that for every $a \in F$, there is an n in the copy of $\mathbb{N}$ with $n > a$) imply Dedekind completeness.

Exercise 4 *Show directly that $\mathbb{R}$ has the Archimedean property.*

Exercise 5 *Show directly that $\mathbb{R}$ is metrically complete.*

Gunning does these two exercises in the Appendix to section 2.2.