Assume u:I→$\mathbb{R}$ is a monotone non-decreasing function defined on an interval I.
Let us consider I = {n ∈ $\mathbb{R}$: a ≤ n ≤ b}
Let c ∈

$$a, b$$

. Then we can define the function:

$$\text{f(c)} = \lim_{x \to c^+} u(x) \text{ - } \lim_{x \to c^-} u(x)$$

Note that f(c) is always positive since u is non-decreasing
Thus, f(c) ≤ u(b) - u(a) Further, the sum of all f(c) ≤ u(b) - u(a)
Then we can define the set

F(n) = {the set of all f(c) ≥ 1/n} for n ∈ $\mathbb{N}$
and
F = {the set of all f(c)}

Since the sum of all f(c) must be ≤ u(b) - u(a), the set F(n) is finite for all n.

Lemma:
WTS: The union of countably many countable sets is countable.
Let each of the countable sets be denoted as S_n where n ∈ $\mathbb{N}$
We can create a 1-1 relation between the union of S_n and $\mathbb{N}$ as follows:

Let s_{nm} represent the m^{th} element from S_n
Now let 1→s_{11}, 2→s_{12}, 3→s_{21}, 4→s_{13}, etc.

Thus, since we have an inductive relationship between the natural numbers and the union of countably many countable sets, we can say that they have the same cardinality. □

Therefore, since F is the union of all F(n) where each F(n) is countable, then F is the union of countably many countable sets.
Thus, F is countable. □