Solution of Starter 5 and Problem 7 of Facts from Analysis

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1 Introduction

Starter 5 Define what it means for a subset of $\mathbb{R}$ to be **open**.

Problem 7 If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continous function and $U \subset \mathbb{R}$ is an open set, show $f^{-1}(U)$ is an open set.

2 Solution

Let $p_0 \in \mathbb{R}$ and $\epsilon > 0$ a real number. Then the $\epsilon -$ **neighborhood** of point p_0 with a metric $\rho(x, y)$ is a subset of $\mathbb{R}$ is defined as:

$$N_{p_0}(\epsilon) = \{p \in \mathbb{R} : \rho(p_0, p) < \epsilon\} \tag{1}$$

In some analysis textbooks this named as an *open ball* in $\mathbb{R}$ of center p_0 and radius ϵ.

An **open set** in $\mathbb{R}$ is defined in terms of these $\epsilon -$ **neighborhoods** as a subset of $U \subset \mathbb{R}$ such that for each $p_0 \in U$, there is an $\epsilon > 0$ such that $N_{p_0}(\epsilon) \subset U$. In other words, a subset of $\mathbb{R}$ is **open** if, for each p in the subset contains some open ball of center p.

Now, we can solve Problem 7. By assuming the continuity of f, we want to show that if $U \subset \mathbb{R}$ is open, then $f^{-1}(U)$ is open as well. We can explicitly write $f^{-1}(U)$ as:

$$f^{-1}(U) = \{p \in E : f(p) \in U\} \tag{2}$$

Now, let $p_0 \in f^{-1}(U)$, so that we have $f(p_0) \in U$. Note that U is open, then it contains some open ball with center $f(p_0)$ and radius $\epsilon > 0$. Also, notice that there exists a $\delta > 0$, such that if $p \in \mathbb{R}$ and $\rho(p, p_0) < \delta$, then $\rho(f(p), f(p_0)) < \epsilon$. That is because f is continuous at p_0. Intuitively, when p is contained in an open ball at the center of p_0 with radius δ, then $f(p)$ is contained in the open ball with a center of $f(p_0)$ and radius ϵ. Note that, this means that $f(p) \in U$. By this, we have $p \in f^{-1}(U) \subset \mathbb{R}$. Then, $f^{-1}(U)$ indeed contains the open ball in $\mathbb{R}$ of center p_0 and radius δ. Finally, this can be generalized to any element of $f^{-1}(U)$ since p_0 was chosen arbitrarily. Then, the set $f^{-1}(U) = \{p \in \mathbb{R} : f(p) \in U\}$ is an open set.