Solution of Problem 8 of Assignment 4A

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1 Introduction

We are asked to show that $d(x, y) = ||x-y||$ defines a distance function on any normed vector space. This is called the **norm induced metric**. Thus, every normed space is a metric space.

For this problem, we have to show that $d(x, y) = ||x - y||$ satisfies the properties of metric which are

- Positive Definite: $d(x, y) = 0$ if and only if $x = y$
- Symmetry: $d(x, y) = d(y, x) \; \forall \; x, y \in X$
- Triangle Inequality: $d(x, y) \leq d(x, z) + d(z, y) \; \forall \; x, y, z \in X$

where X is a metric space.

In order to do this, we have to use the properties of the normed vector space which are

- positivity: $||\mathbf{x}|| \geq 0$ and $||\mathbf{x}|| = 0$ if and only if $\mathbf{x} = 0$
- homogeneity: $||c\mathbf{x}|| = |c|||x||$, for any $c \in \mathbb{R}$
- triangle inequality: $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$

where $\mathbf{x}$ is an element of the vector space V and $||\cdot||$ is defined as a mapping $V \longrightarrow \mathbb{R}$.

2 Solution

Positive Definiteness:

($\longrightarrow$) If $x = y$, then $d(x, y) = d(x, x) = ||x - x|| = ||0|| = 0$
($\longleftarrow$) If $d(x, y) = 0$ then, $||x - y|| = 0$. By the positivity property of the norm, $||x - y|| \longrightarrow x - y = 0 \longrightarrow x = y$

Symmetry :

$$\begin{aligned} d(x, y) &= ||x - y|| \\ &= || - (y - x)|| \\ &= | - 1|||y - x|| \text{ by the homogeneity property} \\ &= d(y, x) \end{aligned} \tag{1}$$

Triangle Inequality

$d(x, y) = ||x - y|| = ||x - z + z - y|| \leq ||x - z|| + ||z - y|| = d(x, z) + d(z, y)$

Therefore, $d(x, y) = ||x - y||$ defines a distance function on any normed vector space.