Solution of Group 1 Problem 1 from Section 1.3 of Gunning

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**1 Introduction**

We are asked to solve the following problem from Section 1.3 of Gunning

If , are linear subspaces of a vector space *V* over a field *F*, which of the following subsets of *V* are also linear subspaces and why:

(i) (ii) (iii) (iv) .

Claim: (i) and (iv) are linear subspaces of *V* over *F* while (ii) and (iii) are not.

**2 Solution**

(i) We will show that given the linear subspaces and , it follows that the intersection of and is also a linear subspace of *V* over *F*. Note that we must satisfy two requirements for a set *W* to be a linear subspace of *V*:

(1) whenever ,

(2) whenever and .

Let . Then, and . So, , , , and . Since and and is a linear subspace, then . Also, since and and is a linear subspace, then . Now, since and , then . Thus, condition (1) is satisfied. Now, consider some and . Since , then and . Since and are both linear subspaces, it follows that and . Thus, and condition (2) is satisfied. Therefore, is a linear subspace of *V* over *F*.

(ii) We will show that given the linear subspaces and , it follows that the union of and is NOT a linear subspace of *V* over *F*. We will show this by giving a counterexample of a and which do not satisfy condition (1) of being a linear subspace. Consider and . Let and . Then, and . It follows that and . However, is not an element of because (1,1) is not an element of nor . Therefore, condition (1) is not satisfied and the union of two linear subspaces is not necessarily a linear subspace.

(iii) We will show that given the linear subspaces and , it follows that the difference of and , , is NOT a linear subspace of *V* over *F*. We will show this by giving a counterexample of a and which do not satisfy condition (1) of being a linear subspace. Consider and . Let and . Then, we have that . Note that (0,0) is not an element of . Since and (1,0) is not an element of , then . Similarly, . Since and , then should be an element of . However, we have already shown that (0,0) is not an element of . Therefore, is not a linear subspace.

(iv) We will show that given the linear subspaces and , it follows that the sum of and is also a linear subspace of *V* over *F*. For clarity, note that the definition of the sum of two vector spaces is as follows:

and

Let , that is, where and . Also, let , that is, where and . Then, which is equivalent to since vector addition is commutative and associative in vector spaces. Since and and is a linear subspace, then . Similarly, . Since, and , then , so . Thus, condition (1) is satisfied. Now, let , that is, where and . Also, let . Then, which is equivalent to since scalar multiplication in vector spaces satisfies the distributive law. Since and is a linear subspace, then . Similarly, . Since and , then . It follows that . Thus, condition (2) is satisfied. Therefore, is a linear subspace.