Solution of Group 1 Problem 1 from Section 1.3 of Gunning

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**1 Introduction**

We are asked to solve the following problem from Section 1.3 of Gunning

If $W\_{1}$, $W\_{2}$ are linear subspaces of a vector space *V* over a field *F*, which of the following subsets of *V* are also linear subspaces and why:

(i) $W\_{1}∩W\_{2}$ (ii) $W\_{1}∪W\_{2}$ (iii) $W\_{1} W\_{2}$ (iv) $W\_{1}+W\_{2}$.

Claim: (i) and (iv) are linear subspaces of *V* over *F* while (ii) and (iii) are not.

**2 Solution**

(i) We will show that given the linear subspaces $W\_{1}$and $W\_{2}$, it follows that the intersection of $W\_{1}$and $W\_{2}$ is also a linear subspace of *V* over *F*. Note that we must satisfy two requirements for a set *W* to be a linear subspace of *V*:

(1) $w\_{1}+w\_{2}\in W$ whenever $w\_{1},w\_{2}\in W$,

(2) $aw\in W$ whenever $a\in F$ and $w\in W$.

Let $w\_{1},w\_{2}\in W\_{1}∩W\_{2}$. Then, $w\_{1}\in W\_{1}∩W\_{2}$ and $w\_{2}\in W\_{1}∩W\_{2}$. So, $w\_{1}\in W\_{1}$, $w\_{1}\in W\_{2}$, $w\_{2}\in W\_{1}$, and $w\_{2}\in W\_{2}$. Since $w\_{1}\in W\_{1}$ and $w\_{2}\in W\_{1}$ and $W\_{1}$ is a linear subspace, then $w\_{1}+w\_{2}\in W\_{1}$. Also, since $w\_{1}\in W\_{2}$ and $w\_{2}\in W\_{2}$ and $W\_{2}$ is a linear subspace, then $w\_{1}+w\_{2}\in W\_{2}$. Now, since $w\_{1}+w\_{2}\in W\_{1}$ and $w\_{1}+w\_{2}\in W\_{2}$, then $w\_{1}+w\_{2}\in W\_{1}∩W\_{2}$. Thus, condition (1) is satisfied. Now, consider some $a\in F$ and $w\in W\_{1}∩W\_{2}$. Since $w\in W\_{1}∩W\_{2}$, then $w\in W\_{1}$ and $w\in W\_{2}$. Since $W\_{1}$ and $W\_{2}$ are both linear subspaces, it follows that $aw\in W\_{1}$ and $aw\in W\_{2}$. Thus, $aw\in W\_{1}∩W\_{2}$ and condition (2) is satisfied. Therefore, $W\_{1}∩W\_{2}$ is a linear subspace of *V* over *F*.

(ii) We will show that given the linear subspaces $W\_{1}$and $W\_{2}$, it follows that the union of $W\_{1}$and $W\_{2}$ is NOT a linear subspace of *V* over *F*. We will show this by giving a counterexample of a $W\_{1}$and $W\_{2}$ which do not satisfy condition (1) of being a linear subspace. Consider $V=R^{2}$ and $F=R$. Let $W\_{1}=\{\left(k,0\right)∨k\in R\}$ and $W\_{2}=\{\left(0,k\right)∨k\in R\}$. Then, $\left(1,0\right)\in W\_{1}$ and $\left(0,1\right)\in W\_{2}$. It follows that $\left(1,0\right)\in W\_{1}∪W\_{2}$ and $\left(0,1\right)\in W\_{1}∪W\_{2}$. However, $\left(1,0\right)+\left(0,1\right)=\left(1,1\right)$ is not an element of $W\_{1}∪W\_{2}$ because (1,1) is not an element of $W\_{1}$ nor $W\_{2}$. Therefore, condition (1) is not satisfied and the union of two linear subspaces is not necessarily a linear subspace.

(iii) We will show that given the linear subspaces $W\_{1}$and $W\_{2}$, it follows that the difference of $W\_{1}$and $W\_{2}$, $W\_{1} W\_{2}$, is NOT a linear subspace of *V* over *F*. We will show this by giving a counterexample of a $W\_{1}$and $W\_{2}$ which do not satisfy condition (1) of being a linear subspace. Consider $V=R^{2}$ and $F=R$. Let $W\_{1}=\{\left(k,0\right)∨k\in R\}$ and $W\_{2}=\{\left(0,k\right)∨k\in R\}$. Then, we have that $W\_{1} W\_{2}=\{\left(k,0\right)∨k\in R−\{0\}\}$. Note that (0,0) is not an element of $W\_{1} W\_{2}$. Since $\left(1,0\right)\in W\_{1}$ and (1,0) is not an element of $W\_{2}$, then $\left(1,0\right)\in W\_{1} W\_{2}$. Similarly, $\left(−1,0\right)\in W\_{1} W\_{2}$. Since $\left(1,0\right)\in W\_{1} W\_{2}$ and $\left(−1,0\right)\in W\_{1} W\_{2}$, then $\left(1,0\right)+\left(−1,0\right)=\left(0,0\right)$ should be an element of $W\_{1} W\_{2}$. However, we have already shown that (0,0) is not an element of $W\_{1} W\_{2}$. Therefore, $W\_{1} W\_{2}$ is not a linear subspace.

(iv) We will show that given the linear subspaces $W\_{1}$and $W\_{2}$, it follows that the sum of $W\_{1}$and $W\_{2}$ is also a linear subspace of *V* over *F*. For clarity, note that the definition of the sum of two vector spaces is as follows:

$W\_{1}+W\_{2}=\{w\_{1}+w\_{2}∨w\_{1}\in W\_{1}$ and $w\_{2}\in W\_{2}\}$

Let $u\in W\_{1}+W\_{2}$, that is, $u=u\_{1}+u\_{2}$ where $u\_{1}\in W\_{1}$ and $u\_{2}\in W\_{2}$. Also, let $v\in W\_{1}+W\_{2}$, that is, $v=v\_{1}+v\_{2}$ where $v\_{1}\in W\_{1}$ and $v\_{2}\in W\_{2}$. Then, $u+v=\left(u\_{1}+u\_{2}\right)+\left(v\_{1}+v\_{2}\right)$ which is equivalent to $\left(u\_{1}+v\_{1}\right)+\left(u\_{2}+v\_{2}\right)$ since vector addition is commutative and associative in vector spaces. Since $u\_{1}\in W\_{1}$ and $v\_{1}\in W\_{1}$ and $W\_{1}$ is a linear subspace, then $u\_{1}+v\_{1}\in W\_{1}$. Similarly, $u\_{2}+v\_{2}\in W\_{2}$. Since, $u\_{1}+v\_{1}\in W\_{1}$ and $u\_{2}+v\_{2}\in W\_{2}$, then $\left(u\_{1}+v\_{1}\right)+\left(u\_{2}+v\_{2}\right)\in W\_{1}+W\_{2}$, so $u+v\in W\_{1}+W\_{2}$. Thus, condition (1) is satisfied. Now, let $u\in W\_{1}+W\_{2}$, that is, $u=u\_{1}+u\_{2}$ where $u\_{1}\in W\_{1}$ and $u\_{2}\in W\_{2}$. Also, let $a\in F$. Then, $au=a\left(u\_{1}+u\_{2}\right)$ which is equivalent to $au\_{1}+au\_{2}$ since scalar multiplication in vector spaces satisfies the distributive law. Since $u\_{1}\in W\_{1}$ and $W\_{1}$ is a linear subspace, then $au\_{1}\in W\_{1}$. Similarly, $au\_{2}\in W\_{2}$. Since $au\_{1}\in W\_{1}$ and $au\_{2}\in W\_{2}$, then $au\_{1}+au\_{2}\in W\_{1}+W\_{2}$. It follows that $au=a\left(u\_{1}+u\_{2}\right)\in W\_{1}+W\_{2}$. Thus, condition (2) is satisfied. Therefore, $W\_{1}+W\_{2}$ is a linear subspace. $∎$