Math 4317, Exam 3: Analysis (practice) Name and section: ____________________

1. (25 points) Find the distance from $A = \{(x, y) : x > 0,\ |y| \le \sin^2 x\}$ to $B = \{(-j, 0) : j \in \mathbb{N}\}$

Solution: Given any $(x, y) \in A$ and $(-j, 0) \in B$,

$$|(x, y) - (-j, 0)| = \sqrt{(x + j)^2 + y^2} \ge \sqrt{(x + j)^2} = x + j > 1$$

since $x > 0$ and $j \ge 1$. On the other hand, taking $j = 1$ and $(1/n, 0) \in A$, we find

$$|(1/n, 0) - (-1, 0))| = 1 + 1/n \to 1 \quad \text{as } n \to \infty.$$

Therefore the distance from A to B is 1.

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2. (25 points) Define the distance between two sets in $\mathbb{R}^n$.

Prove that if A and B are compact sets in $\mathbb{R}^n$, then there is some $a \in A$ and some $b \in B$ with

$$\operatorname{dist}(A, B) = |a - b|.$$

Solution: Given two nonempty sets A and B,

$$\operatorname{dist}(A, B) = \inf\{|a - b| : a \in A,\ b \in B\}.$$

Let $a_j \in A$ and $b_j \in B$ with $|a_j - b_j| \to \operatorname{dist}(A, B)$. Such points exist by the definition of infemum. Since A and B are compact sets, they are bounded. Therefore, $\{a_j\}$ and $\{b_j\}$ are bounded sets as well. We may therefore take a convergent subsequence of $\{a_j\}$. From the corresponding subsequence of $\{b_j\}$ we may also extract a second convergent subsequenct. In this way, we obtain a subsequence $\{a_{j_k}\}$ and the corresponding subsequence $\{b_{j_k}\}$ with the first subsequence converging to a point $a_0 \in \mathbb{R}^n$ and the second converging to a point $b_0 \in \mathbb{R}^n$. Since A and B are closed, we see that $a_0 \in A$ and $b_0 \in B$.

Finally,

$$\begin{aligned} \operatorname{dist}(A, B) &= \lim_{j \to \infty} |a_{j_k} - b_{j_k}| \\ &= |a_0 - b_0| \end{aligned}$$

since $||a_{j_k} - b_{j_k}| - |a_0 - b_0|| \leq |a_{j_k} - b_{j_k} - (a_0 - b_0)| \leq |a_{j_k} - a_0| + |b_{j_k} - b_0| \to 0$ as $k \to \infty$.

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3. (25 points) (Lemma 15.2) Define *convergence* for a sequence of points in $\mathbb{R}^n$.

Prove that if a sequence $\{p_j\}_{j=1}^{\infty}$ converges to a point q, then any subsequence of $\{p_j\}$ also converges to q.

Solution: A sequence $\{p_j\} \subset \mathbb{R}^n$ *converges* to $q \in \mathbb{R}^n$ if for any $\epsilon > 0$, there is some $N > 0$ for which

$$j > N \qquad \Longrightarrow \qquad p_j \in B_\epsilon(q).$$

We first note that the indices j_k of any subsequence satisfy $j_k \geq k$ for all k. (This follows from an easy induction.)

Now, given a subsequence $\{p_{j_k}\}$ of the convergent sequence $\{p_j\}$, there is some N such that

$$k > N \qquad \Longrightarrow \qquad p_k \in B_\epsilon(q).$$

Consequently,

$$k > N \qquad \Longrightarrow \qquad j_k \geq k > N \qquad \Longrightarrow \qquad p_{j_k} \in B_\epsilon(q).$$

This means $p_{j_k} \to q$ as $k \to \infty$.

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4. (25 points) (16I(b)) Define what it means for a sequence to be *Cauchy*.

Show that the sequence $\{(j+1)/j\}$ is Cauchy.

Solution: $\{p_j\}$ is *Cauchy* if for any $\epsilon > 0$, there is some N such that when $j, k > N$ there holds $|p_j - p_k| < \epsilon$.

Let $\epsilon > 0$. Note that

$$\frac{j+1}{j} - \frac{k+1}{k} = \frac{1}{j} - \frac{1}{k}.$$

Thus, if $N > 2/\epsilon$ and $j, k > N$, then

$$\left| \frac{j+1}{j} - \frac{k+1}{k} \right| \leq \frac{1}{j} + \frac{1}{k} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$