Math 4317, Exam 2 (practice) Name and section: ________________

1. (25 points) Define the term *compact*.

Prove directly from the definitions (without using the Heine-Borel Theorem) that a closed subset of a compact set is compact.

Solution: A set K is compact if any open cover of K has a finite subcover.

Let K be a compact set and let C be a closed subset of K. Then notice that $U_0 = C^c$ is an open set. Let $\{U_\alpha\}_{\alpha\in\Gamma}$ be any open cover of C. Then $\{U_\alpha\} \cup \{U_0\}$ is an open cover of K. Since K is compact, this cover has a finite subcover:

$$\{U_0, U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_m}\}.$$

We claim that $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_m}\}$ is a finite cover of C. In fact, if $x \in C$, then $x \in K\backslash U_0$. Therefore x must be in $\cup U_{\alpha_j}$. Therefore, $C \subset \cup U_{\alpha_j}$, and $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_m}\}$ is a cover. It is therefore a finite open subcover of C, and C is compact.

Name and section: ____________________

2. (25 points) Define what it means for a function $f : X \to \tilde{X}$ to be *continuous at a point* $p_0 \in X$ where X and $\tilde{X}$ are metric spaces with distances d and $\tilde{d}$ respectively.

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = \sqrt[4]{x^2 + y^2}$$

where $\mathbb{R}^2$ and $\mathbb{R}$ are taken with the usual Euclidean metrics. Prove that f is continuous at 0.

Solution:

f is *continuous at a point* $p_0 \in X$ if for any $\epsilon > 0$, there is some $\delta > 0$ such that

$$d(p, p_0) < \delta \qquad \text{implies} \qquad \tilde{d}(f(p), f(p_0)) < \epsilon.$$

Let $\epsilon > 0$. Set $\delta = \epsilon^2$. If

$$d((x, y), (0, 0)) = \sqrt{x^2 + y^2} < \delta,$$

then

$$|f(x, y) - f(0, 0)| = |\sqrt[4]{x^2 + y^2}| < \sqrt{\delta} = \epsilon.$$

This means that f is continuous at $(0, 0)$.

Name and section: ____________________

3. (25 points) Given a fixed point p_0 in a metric space X with distance d, consider the function $f : X \to \mathbb{R}$ by $f(x) = d(p_0, x)$. Show that f is continuous on X.

Solution: Let $q_0 \in X$ and ϵ be fixed. Let $\delta = \epsilon$ and assume $d(x, q_0) < \delta$. We first note that

$$|f(x) - f(q_0)| = |d(p_0, x) - d(p_0, q_0)|.$$

On the one hand, $d(p_0, x) - d(p_0, q_0) \leq d(q_0, x)$ by the triangle inequality. On the other hand, $d(p_0, q_0) - d(p_0, x) \geq -d(q_0, x)$ (also by the triangle inequality). Therefore,

$$|d(p_0, x) - d(p_0, q_0)| \leq d(q_0, x) < \delta = \epsilon,$$

and we have shown f is continuous at q_0.

Name and section: ________________

4. (25 points) Given a sequence of real valued functions $\{f_j\}_{j=1}^{\infty}$ with common domain a metric space X, define what it means for these functions to *converge uniformly.*

Give an example of a sequence of functions $f_j \in C^0[0,1]$ which converges pointwise at every point, but does not converge uniformly.

Solution: The sequence converges uniformly, if there is a function $f : X \to \mathbb{R}$ such that for any $\epsilon > 0$, there is some N such that $j > N$ implies $|f_j(x) = f(x)| < \epsilon$ for every $x \in X$.

The sequence of functions determined by $f_j(x) = x^j$ satisfies the desired conditions. For each $a < 1$, we find $f_j(a) \to 0$, and $f_j(1) \equiv 1$. Thus, we have pointwise convergence at every $a \in [0,1]$ to the function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0, & x < 1 \\ 1, & x = 1. \end{cases}$$

On the other hand, by the intermediate value theorem, there points a_j for every j with $f_j(a_j) = 1/2$. It is therefore clear that

$$\max_{a \in [0,1]} |f_j(a) - f(a)| \geq 1/2,$$

so f_j does not converge uniformly to f. (Note: In fact, the value 1/2 may be replaced with any positive value less than 1.)