Math 4317, Exam 1 (practice) Name and section: ____________________

1. **(i)** State precisely the definition of an algebraic *group*.

(ii) Are the integers a *field*? (Explain your reasoning carefully.)

(25 points) **Solution:**

(i) A *group* is a set G together with an operation $* : G \times G \to G$ satisfying the following:

G1: $(a * b) * c = a * (b * c)$ for all elements $a, b, c \in G$. This is called the associative property.

G2: There is a special element e (called the identity) with the property that $a * e = a = e * a$ for all $a \in G$.

G3: For each $a \in G$, there is an element $b \in G$ such that $a * b = e = b * a$. The element b is called an *inverse* of a.

(ii) The integers are not a field because it is required that the nonzero elements of a field form a group under multiplication. We know, however, that 2 is a nonzero integer, and there is no integer k for which $2k = 1$. This means, 2 has no multiplicative inverse, so the nonzero integers are not a group, and the integers, consequently, are not a field.

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2. (25 points) Give examples of sets of real numbers A, B, and C with the following properties:

(i) $\sup A \in A$.

(ii) $\sup B \in \mathbb{R} \backslash B$.

(iii) $\sup C \notin \mathbb{R}$.

Solution:

(i) $A = \{0\}$. In this case, $\sup A = 0 \in A$.

(ii) $B = \{1 - 1/j : j \in \mathbb{N}\}$. In this case, $\sup B = 1 \in \mathbb{R} \backslash B$.

(iii) $C = \mathbb{N}$. In this case, $\sup C = +\infty \notin \mathbb{R}$.

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3. (25 points) (6C) Define *bounded above.*

Define *upper bound.*

Define *supremum.*

Solution:

1. Given a set A of real numbers, we say A is *bounded above* if there is a real number M with $M \geq x$ for every $x \in A$.
2. Given a set A of real numbers which is bounded above, we say a real number M is an *upper bound* for A if $x \leq M$ for every $x \in A$.
3. Given a nonempty set A of real numbers which is bounded above, a number M is the *supremum of* A (or the *least upper bound* of A), if M is an upper bound for A and if $B \in \mathbb{R}$ is any other upper bound for A, then $B \geq M$.

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4. **(i)** Define what it means for a set in a metric space to be *open*.

(ii) For this question, let us define a *closed* set to be one whose complement is open. Using your definition from part (i), show that a finite union of closed sets is closed.

(25 points) **Solution:**

(i) A set U in a metric space X is *open* if whenever $x \in U$, there is some $r > 0$ such that

$$B_r(x) = \{\xi \in X : d(\xi, x) < r\} \subset U$$

where $d : X \times X \to [0, \infty)$ is the distance function for X.

(ii) Let $C_1, C_2, \ldots, C_m$ be closed sets in a metric space X. If $x \in (\cup C_j)^c = \cap C_j^c$, then there are numbers $r_1, \ldots, r_m$, all positive, such that

$$B_{r_j}(x) \subset C_j^c \quad \text{for } j = 1, \ldots, m.$$

(This is because each C_j^c is open.)

Let $r = \min\{r_1, \ldots, r_m\}$. Notice that $r > 0$. Furthermore,

$$B_r(x) \subset B_{r_j}(x) \subset C_j^c \quad \text{for } j = 1, \ldots, m.$$

Therefore, $B_r(x) \subset \cap C_j^c = (\cup C_j)^c$. This means, $(\cup C_j)^c$ is open, and that means $\cup C_j$ is closed. $\square$