Math 4317, Final Exam: Analysis (practice)

1. (20 points) Prove or disprove: The intersection of open sets is open. (Fully justify your answer.)

Solution: Disproof: Each of the intervals $(0, 1 + 1/j)$ for $j = 1, 2, \ldots$ is open, but

$$\cap_{j=1}^{\infty}(0, 1 + 1/j) = (0, 1]$$

is not open because there is no open ball $B_r(1)$ with positive radius r and center $x = 1$ which lies entirely in the intersection.

2. (a) (10 points) Define the term *compact*.

(b) (10 points) Prove directly from the definitions (without using the Heine-Borel Theorem) that a compact set is closed.

Solution:

(a) A set K is compact if any open cover of K has a finite subcover.

(b) We need to show the complement of K is open. Let K be a compact set and let x be in the complement of K. Then notice that $\{\mathbb{R}^n \backslash \overline{B_r(x)}\}_{r>0}$ is an open cover of K. Since K is compact, this cover has a finite subcover:

$$\{\mathbb{R}^n \backslash \overline{B_{r_1}(x)}, \mathbb{R}^n \backslash \overline{B_{r_2}(x)}, \ldots, \mathbb{R}^n \backslash \overline{B_{r_k}(x)}\}.$$

Taking $r = \min\{r_1, \ldots, r_k\} > 0$, we find that $K \subset \mathbb{R}^n \backslash \overline{B_r(x)}$. Thus, $B_r(x) \subset K^c$, and it follows that the complement K^c is open. Thus, K is closed.

3. Problems 3 and 4 involve the sequence of functions $f_j : (0,1) \to \mathbb{R}$ given by $f_j(x) = 1/(x + 1/j)$ for $j = 1, 2, \ldots$.

   (a) (10 points) Define the term *uniformly continuous.*

   (b) (10 points) Prove or disprove: Each function f_j is uniformly continuous.

Solution:

(a) Given a function $f : A \to \mathbb{R}^m$ with $A \subset \mathbb{R}^n$, we say f is *uniformly continuous* if given $\epsilon > 0$, there is some $\delta > 0$ such that

$$|f(x_2) - f(x_1)| < \epsilon \quad \text{whenever} \quad x_1, x_2 \in A \text{ with } |x_2 - x_1| < \delta.$$

(b) Proof: Let j be one of the functions in question. Given $\epsilon > 0$, let $\delta = \epsilon/j^2$ and note that $|x+1/j|, |x_0+1/j| \geq 1/j$ for any $x, x_0 \in (0,1)$, so $|(x+1/j)(x_0+1/j)| \leq 1/j^2$. It follows that whenever x and x_0 are in $A = (0,1)$ with $|x - x_0| < \delta$, then

$$\begin{aligned}
|f_j(x) - f_j(x_0)| &= |1/(x + 1/j) - 1/(x_0 + 1/j)| \\
&= |x_0 - x|/|(x + 1/j)(x_0 + 1/j)| \\
&\leq j^2|x - x_0| \\
&< \epsilon.
\end{aligned}$$

This shows that f_j is uniformly continuous.

4. (a) (5 points) Define what it means for a sequence of functions $g_j : (0,1) \to \mathbb{R}$ for $j = 1, 2, \ldots$ to *converge pointwise* to a given function $g : (0,1) \to \mathbb{R}$.

   (b) (5 points) Show that the sequence of functions $\{f_j\}_{j=1}^{\infty}$ from problem 4 converges pointwise to some function f.

   (c) (5 points) Define what it means for a sequence of functions $g_j : (0,1) \to \mathbb{R}$ for $j = 1, 2, \ldots$ to *converge uniformly* to a given function $g : (0,1) \to \mathbb{R}$.

   (d) (5 points) Prove or disprove: the sequence of functions $\{f_j\}_{j=1}^{\infty}$ from problem 4 converges uniformly to f.

Solution:

(a) A sequence of functions $\{g_j\}$ *converges pointwise* to a function g if for each fixed x, and each $\epsilon > 0$, there is some N such that

$$j > N \qquad \text{implies} \qquad |g_j(x) - g(x)| < \epsilon.$$

(b) The limit function is $f(x) = 1/x$. To see this, fix any $x > 0$. Then

$$|f_j(x) - f(x)| = |1/(x + 1/j) - 1/x| = \frac{1/j}{x(x + 1/j)} < \frac{1}{jx^2}.$$

Thus, if ϵ is any positive number and $N > 1/(\epsilon x^2)$,

$$j > N \qquad \text{implies} \qquad |f_j(x) - f(x)| < \epsilon.$$

(c) Such a sequence converges *uniformly* if there is some N such that

$$j > N \qquad \text{implies} \qquad |g_j(x) - g(x)| < \epsilon \text{ (for all } x).$$

(d) Disproof: The sequence $\{f_j\}$ does not converge uniformly to $1/x$. To see this, take $\epsilon_0 = 1$. Given any N, we need to find some x and some $j > N$ with $|f_j(x) - f(x)| > 1$. Using the equalities in the computation of part (b) above, we see that for any x and j with $x < 1/j$, we have $|f_j(x) - f(x)| > 1/(2x)$. Thus, fixing any $j > N$, we can take $x < \min\{1/j, 1/2\}$, and we have $|f_j(x) - f(x)| > 1 = \epsilon_0$.

5. (a) (10 points) Give a correct statement of the mean value theorem.

(b) (10 points) Use the mean value theorem to prove: If $f, g \in C^0[a, b]$ and the derivatives $f', g' : (a, b) \to \mathbb{R}$ exist with $f' \equiv g'$, then there is some constant $c \in \mathbb{R}$ such that $f(x) = g(x) + c$ for $x \in [a, b]$.

Solution:

(a) MVT: If $\phi \in C^0[a, b]$ and the derivative $\phi' : (a, b) \to \mathbb{R}$ exists, then there is some $\xi \in (a, b)$ such that

$$\phi'(\xi) = \frac{\phi(b) - \phi(a)}{b - a}.$$

(b) Consider the function $\phi = f - g$. This function satisfies the hypotheses of the mean value theorem on every interval $[\alpha, \beta] \subset [a, b]$, and $\phi' \equiv 0$. Consequently, for some $\xi \in (a, b)$,

$$\frac{f(\beta) - g(\beta) - [f(\alpha) - g(\alpha)]}{\beta - \alpha} = \phi'(\xi) = 0.$$

Therefore, $\phi(x) = f(x) - g(x) \equiv c$ is constant. This means $f(x) = g(x) + c$.

6. (a) (10 points) Define the Riemann integral of a function $f : [a, b] \to \mathbb{R}$.

(b) (10 points) Prove the integral you have defined, if well-defined, is unique.

Solution:

(a) Given a function $f : [a, b] \to \mathbb{R}$, the number I is the integral of f over the interval $[a, b]$ if the following holds:

For any $\epsilon > 0$, there is some $\delta > 0$ such that

$$\left| \sum_{j=0}^{k-1} f(x_j^*)(x_{j+1} - x_j) - I \right| < \epsilon$$

whenever $\mathcal{P} : a = x_0 < x_1 < \cdots < x_k = b$ is a partitition of $[a, b]$ with

$$\|\mathcal{P}\| = \max_j (x_{j+1} - x_j) < \delta$$

and $x_j^* \in [x_j, x_{j+1}]$ for $j = 0, 1, \ldots, k - 1$.

(b) Let I and J be two numbers which both satisfy the definition of

$$\int_a^b f(x)dx.$$

Let $\epsilon > 0$. Take a particular partition with small enough norm so that

$$\left| \sum_{j=0}^{k-1} f(x_j)(x_{j+1} - x_j) - I \right| < \epsilon/2$$

and

$$\left| \sum_{j=0}^{k-1} f(x_j)(x_{j+1} - x_j) - J \right| < \epsilon/2.$$

We then have by the triangle inequality

$$|I - J| \le \left| I - \sum_{j=0}^{k-1} f(x_j)(x_{j+1} - x_j) \right| + \left| \sum_{j=0}^{k-1} f(x_j)(x_{j+1} - x_j) - J \right| < \epsilon.$$