Open Intervals and Open Sets

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February 25, 2020

1 Open Intervals

Given that at **interval** is a subset I of the real numbers $\mathbb{R}$ for which

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\} \subset I \qquad \text{whenever } a, b \in I,$$

we have shown there are ten different kinds of intervals. Five of these kinds are referred to informally as open intervals:

$$\phi,\ \mathbb{R} = (-\infty, \infty), \tag{1}$$

$$(a, \infty) = \{x \in \mathbb{R} : a < x\} \qquad \text{where } a \in \mathbb{R}, \tag{2}$$

$$(-\infty, b) = \{x \in \mathbb{R} : x < b\} \qquad \text{where } b \in \mathbb{R}, \text{ and} \tag{3}$$

$$(a, b) = \{x \in \mathbb{R} : a < x < b\} \qquad \text{where } a, b \in \mathbb{R}. \tag{4}$$

Definition 1 *We say a subset $A \subset \mathbb{R}$ is* **bounded** *if there is some $R \in \mathbb{R}$ such that*

$$-R < a < R \qquad \textit{for all } a \in A.$$

A set $A \subset \mathbb{R}$ is **unbounded** *if it is not bounded.*

Exercise 1 *Show that a set in $\mathbb{R}$ is bounded if and only if it is bounded above and bounded below.*

Exercise 2 *Among the open intervals only ϕ and (a, b) for $a, b \in \mathbb{R}$ are bounded. Only the type (a, b) for $a, b \in \mathbb{R}$ is* **nonempty and bounded**.

So far, we have only used the term "open interval" to refer to sets of a specific form in $\mathbb{R}$. Continuing this usage, we have the following result:

Theorem 1 **(a)** *Every open interval I is the union of the bounded open intervals (a, b) with $a, b \in \mathbb{R}$ such that $(a, b) \subset I$.*

(b) *Let $\{I_\alpha : \alpha \in \Gamma\}$ be a collection of nonempty bounded open intervals such that $I_\alpha \cap I_\beta \neq \phi$ for $\alpha, \beta \in \Gamma$. Then*

$$\bigcup_{\alpha \in \Gamma} I_\alpha \qquad \text{is an open interval.}$$

(c) *Let $\{I_\alpha : \alpha \in \Gamma\}$ be a collection of nonempty bounded open intervals such that*

$$\text{Given } \alpha, \beta \in \Gamma, \text{ either } I_\alpha \subset I_\beta \text{ or } I_\beta \subset I_\alpha. \tag{5}$$

Then

$$\bigcup_{\alpha \in \Gamma} I_\alpha \qquad \text{is an open interval.}$$

(d) *Every open interval I is the union of* **nested** *bounded open intervals.*

Note that (c) is a corollary of (b). We say a collection $\{I_\alpha : \alpha \in \Gamma\}$ satisfying (5) is **nested**.

Theorem 2 **(a)** *Given any nonempty open interval I,*

$$(\inf I, \sup I) \subset I.$$

(b) *If (a, b) is a nonempty open interval with $(a, b) \subset I$, then*

$$(a, b) \subset (\inf I, \sup I) \subset I.$$

In view of these results, we may make the following definitions:

Definition 2 *Given a nonempty open interval I, the* **largest open interval**

$$(\inf I, \sup I) \subset I$$

is called the **interior** *of I. If $\inf I \in \mathbb{R}$, we call $\inf I$ the* **left endpoint** *of I. If $\sup I \in \mathbb{R}$, we call $\sup I$ the* **right endpoint** *of I.*

2 Open Sets

There is a more general (and more commonly intended) use of the word “open” in “open interval.” In other words, this is what a person really means when he says I is an “open interval.” Or maybe this is what he “should” mean. In any case, the meaning is not special to intervals:

Definition 3 *A set $U \subset \mathbb{R}$ is* **open** *if for each $x \in U$, there is some $r > 0$ such that*

$$(x + r, x - r) \subset U.$$

If we wish to distinguish this condition from something else, we can say **topologically open**.

Exercise 3 *Show that $U \subset \mathbb{R}$ is open if and only if for each $x \in U$, there is some $a \in [-\infty, x)$ and $b \in (x, \infty]$ with $(a, b) \subset U$.*

Theorem 3 **(a)** *Each of the five types of “open intervals” is open.*

(b) *If I is an interval and I is open (i.e., topologically open as a set in $\mathbb{R}$), then I has one of the five forms given in (1)-(4).*

(c) *If I is a nonempty bounded interval and is open, then there exists*

$$a = \inf I \in \mathbb{R} \backslash I \qquad \text{and} \qquad b = \sup I \in \mathbb{R} \backslash I$$

such that

$$I = (a, b).$$

The notion of open sets can be adapted to a great many more contexts than that of subsets of $\mathbb{R}$, and is in many respects an extremely important notion. The **collection of all open subsets** of $\mathbb{R}$ is called the **topology** of $\mathbb{R}$. When considered in reference to the collection $\mathcal{T}$ of all open subsets, the real numbers $\mathbb{R}$ is called a **topological space**.

Exercise 4 *Let $\mathcal{T}$ be the topology on $\mathbb{R}$.*

(a) *Show $\phi, \mathbb{R} \in \mathcal{T}$. (The empty set and the whole space are open.)*

(b) *Show arbitrary unions of open sets are open.*

(c) *Show finite intersections of open sets are open.*

The topology is closed under arbitrary unions and finite intersections.