Remarks on the Construction of Numbers for Analysis

John McCuan

January 7, 2020

A typical undergraduate course in analysis, like this course or the one for which our textbook *An Introduction to Analysis* by Robert Gunning was written, is built on a foundation of particular number systems. These number systems include

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3, \ldots\} \qquad \text{(the natural numbers)}\\
\mathbb{N}_0 &= \{0, 1, 2, 3, \ldots\} \qquad \text{(the natural numbers with zero)}\\
\mathbb{Z} &= \{0, \pm 1, \pm 2, \pm 3, \ldots\} \qquad \text{(the integers)}\\
\mathbb{Q} &= \{p/q : p \in \mathbb{Z}, q \in \mathbb{N}\} \qquad \text{(the rational numbers)}\\
\mathbb{R} &= \{x : -\infty < x < \infty\} \qquad \text{(the real numbers)}\\
\mathbb{C} &= \{x + iy : x, y \in \mathbb{R}\} \qquad \text{(the complex numbers)}\\
&\mathbb{R} \cup \{\infty\} \quad \text{and/or} \quad \mathbb{R} \cup \{\pm\infty\} \qquad \text{(the extended real numbers)}\\
&\mathbb{C} \cup \{\infty\} \qquad \text{(the Riemann sphere or extended complex numbers).}
\end{aligned}$$

Some (if not many) of the properties of each of these sets must be "known" to make progress and understand analysis. For most of you, at this stage in your mathematical education/journey (whatever you want to call it), probably your comfort level with the construction and intracacies of these sets starts to run out somewhere between $\mathbb{Q}$ and $\mathbb{R}$ on this list. Naturally then, extending at least the comfort you have in working with integers and formally considering some properties of any and all of these sets is included in the material of any introductory course in analysis at this level. There is no natural prerequisite course in which this material is covered. It is also worth noting, on the other hand, that there are a great many mysteries about even the prime numbers within the simplest set $\mathbb{N}$ which are not known to anyone, and cannot be covered.

From a historical perspective, after all these sets were in common use both in the practice and teaching of analysis, it was discovered that the rigorous consideration of these sets in analysis can lead to some unsettling questions for which no one had good answers. For example:

> *Is it possible to prove two statements about natural numbers which are mutually contradictory?*
>
> —David Hilbert (1900)

It turns out the answer is very difficult. It is not an unqualified "yes" to say the least.[1] Nevertheless, a great deal of effort and careful thought has been applied to questions like this perhaps culminating in Gödel's incompleteness theorems (1931). In view of the clarifying contribution of this work, which may be classed under the broad heading of *foundations*, and even before it, many presentations of analysis included at least some discussion of the construction of the sets listed above along with their properties. The extent to which this discussion should be pursued is largely a matter of taste and may be said to vary between two extremes represented by two well-known texts. In his 1951 book *Foundations of Mathematical Analysis* Edmund Landau concludes on page 37 (well over one-third of the way through the book in Theorem 95) that for $X, Y, Z \in \mathbb{Q}$, if $X > Y$, then $X + Z > Y + Z$. It takes another thirteen pages until one finds on page 50 the same conclusion for real numbers x, y, and z in Theorem 134, though technically the result is only for certain "Dedekind cuts" x, y, and z which are not identified with positive real numbers until page 69 (another 19 pages and over half way through the book). At the other extreme one finds texts like Robert Bartle's 1964 book *The Elements of Real Analysis* which says simply (on page 22): "We shall assume familiarity with the set of natural numbers" and follows this up on page 27 with the enlightening statement that the construction of $\mathbb{R}$ from $\mathbb{Q}$ is "possible," but will not be presented. To be fair, Bartle does mention in a footnote that the interested reader may find more information about the construction of numbers in the 1960 book *Naive Set Theory* by Paul Halmos and gives, in fact, most of the ideas for the construction of $\mathbb{R}$ from $\mathbb{Q}$ via Dedekind cuts in sections 6 and 7 of Chapter 2. It may be added that most people (and mathematicians too) find the books of Landau and Halmos rather tedious, so while the line must be drawn somewhere, and there is some itching feeling that people like Landau and Halmos seem to have had to

[1] See Timothy Chow's paper *The Consistency of Arithmetic* from 2018 in The Mathematical Intelligencer.

tell students everthing they could about foundations (and I feel a little bit like that myself), the urge is probably best resisted.

The text of Gunning may be said to take something of a middle road facilitated especially in the construction of the reals $\mathbb{R}$ from the rationals $\mathbb{Q}$ (in an Appendix to section 2.2) by the use of somewhat heavy handed abstraction. Namely, he builds up the machinery of metric spaces and abstract Cauchy sequences first and then, rather retrospectively, "defines" the real numbers as equivalence classes of Cauchy sequences in $\mathbb{Q}$ (considered as a metric space). This is in contrast to, and bypasses completely, the rather more direct approach involving Dedikind cuts. The overall presentation may be criticized in several respects, but generally I think it's not too bad. Let's return to the natural numbers and start with some criticism there. Gunning introduces the natural numbers as cardinalities of "formal symbols"

$$\{/\},\ \{/,/\},\ \{/,/,/\},\ldots \tag{1}$$

This **looks like** set notation—it is identical to set notation, according to which $\{/\} = \{/,/\} = \{/,/,/\} = \cdots$—which is, of course, not what he means. From there Gunning basically says all the properties of the Peano Axioms are "evident" from (1). I think this definitely borders on dishonesty. On the other hand, as hinted at by the names of Gödel's theorems, one is going to be forced here to take something on faith. I prefer to be somewhat more honest about that and say:

1. Let's **assume** there is a set.

   You may need to go back and read that assumption again. I'm not saying to assume the existence of any particular set...just some set, say A. Then there is the *specified set*
   $$\phi = \{x \in A : x \neq x\}.$$
   If you look in the book of Halmos, you'll find something like an *axiom of specification,* and we're using that here.[2] This specified set ϕ is called the **empty set**.

2. A minor modification of Gunning's slashfest (1) is how I think of the **natural numbers**:
   $$0 = \phi,\ 1 = \{\phi\},\ 2 = \{0, 1\} = \{\phi, \{\phi\}\},\ldots \tag{2}$$

[2]You'll also find a host of other axioms given by Halmos, of which it can be said "we are using them," but we will not say it. For example, There is an "Axiom of pairing" which says that every two sets are elements of some set. We can take the two sets to both be the empty set and write $\{\phi\}$ with confidence.

The set $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, where each element (number) $n+1$ is a set obtained according to the inductive rule

$$n + 1 = n \cup \{n\},$$

will be **assumed** to exist and satisfy the Peano axioms. Gunning's statement of the Peano axioms seems just fine to me, but I'll state them here just for practice/repetition:

(a) There is a **first element** $0 \in \mathbb{N}_0$.

(b) Every element $n \in \mathbb{N}_0$ has a **successor** $n + 1 \in \mathbb{N}_0$, and successors satisfy

  i. $n + 1 \neq 0$,

  ii. $n + 1 \neq n$, and

  iii. If $n + 1 = m + 1$, then $m = n$.

(c) If $E \subset \mathbb{N}_0$ with $0 \in E$ and we know that

$$n \in E \qquad \Longrightarrow \qquad n + 1 \in E,$$

then $E = \mathbb{N}_0$. This is called the **principle of induction**.

3. Nobody seems to have obtained/proved any two mutually contradictory assertions involving the natural numbers yet...nor have any such statements involving the other sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ constructed from them surfaced. So we'll hope for the best.

4. We'll give a few details of the construction of these other sets in addition to (2) which may serve as a kind of construction of $\mathbb{N}_0 \supset \mathbb{N}$.

As a brief follow-up to Hilbert's question and a somewhat contrarian view of Chow's paper, I think what Hilbert had in mind was starting with some very simple axiom or axioms like our first assumption above (there exists a set). Then you get the empty set, and (somehow) you prove consistency with respect to this (very simple) structure within the structure itself. Part of the problem, I think, was that Hilbert didn't have the formal tools of logic nor understand those were necessary for such a program (now known as Hilbert's program) to be carried out. One way to view what eventually came out in the incompleteness theorems of Gödel is that when you figure out the required logical framework and the set theory, then (first of all) things do not get simpler, they get more complicated and (second) though it is possible to state what it means for the resulting system to be consistent, it is impossible to prove consistency within that system itself. Therefore, you are stuck assuming something—and not just a little something.

To wind up these preliminary comments, I'll list a few other well-known texts you may wish to consult and compare:[3]

1. *Advanced Calculus* by R. Creighton Buck (1956)

2. *Introduction to Analysis* by Maxwell Rosenlicht (1968)

3. *Principles of Mathematical Analysis* by Walter Rudin (1976)

4. *The Theory of Functions of a Real Variable* by E.W. Hobson (1907)

5. *Foundations of Modern Analysis* by Jean Dieudonné (1969)

[3] I'll try to put the first published date; there are likely newer editions. Incidentally, I've read most of all of these and found them very influential.