The Completeness of the Integers

John McCuan

February 25, 2020

1 Max/Min and Sup/Inf

Throughout this discussion, let X be a partially ordered set. We are primarily interested in the examples $X = \mathbb{Z}$ and $X = \mathbb{N}$ which are totally ordered, but we will refer to those sets when they are discussed in particular. Let us first recall some definitions and perhaps introduce some new ones.

Definition 1 *(upper bound) Given a set $A \subset X$, an element $U \in X$ is an* **upper bound** *for A (or an upper bound of A) if*

$$a \le U \qquad \text{for every } a \in A.$$

If there exists an upper bound for A, then we say A is **bounded above**.

Exercise 1 *Define what it means for $\ell \in X$ to be a* **lower bound** *for $A \subset X$ and what it means for A to be* **bounded below**.

Definition 2 *(maximum/minimum) Given $A \subset X$, an element $M \in A$ is a* **maximum element** *of A if*

$$a \le M \qquad \text{for every } a \in A.$$

Similarly, $m \in A$ is a **minimum element** *of A if*

$$m \le a \qquad \text{for every } a \in A.$$

Exercise 2 *If M is a maximum element of A, then M is unique, i.e., M is the only maximum element in A. Similarly, if $m \in A$ is a minimum element, then m is unique.*

In view of the previous exercise, if M is a maximum element of A, we call M *the* maximum element of A and write

$$M = \max A.$$

Similarly, if m is a minimum element of A, we call m *the* minimum element and write

$$m = \min A.$$

Our primary objective is, perhaps, to prove the following result:

Theorem 1 *If $A \subset \mathbb{Z}$ is nonempty and bounded above, then there is a unique element $M \in A$ with*

$$M = \max A.$$

Similarly, If A is bounded below, then there is a unique $m \in A$ with

$$m = \min A.$$

In addition, we will discuss various related concepts and examples. While a maximum or minimum element is unique, this may not be true of an upper or lower bound.

Exercise 3 *Give examples of partially ordered sets X and subsets $A \subset X$ satisfying the following conditions:*

1. *A is bounded above but does not have a unique upper bound.*
2. *A is bounded above and does have a unique upper bound U, but U is not a maximum element for A.*
3. *A is bounded above, does not have a unique upper bound, and has no maximum element.*

Nevertheless, something can be said in general.

Lemma 1 *If $A \subset X$, then the following hold:*

(a) *If $U \in X$ is an upper bound for A and $U \in A$, then $U = \max A$.*

(b) *If $\max A$ exists, then A is bounded above.*

This result seems trivial, but don't sell it short: We will use it in our main proof below. A seemingly more substantial general assertion we will also use is the following:

Lemma 2 *Every* **nonempty finite** *subset of a* **totally** *ordered set has a maximum element and minimum element.*

Proof: Let A be a finite subset of a totally ordered set X. Our proof is by induction on the number of elements in A. If $A = \{a\}$ has only one element, then clearly that element a has $a = \max A = \min A$.

Assume we know any subset $A \subset X$ has a maximum and minimum if $\#A \leq k$. Consider a set S with $\#S = k + 1$. Taking any particular element $a \in S$, we note that $\#S \backslash \{a\} = k$. By the inductive hypothesis, there are elements

$$m = \min S \backslash \{a\} \in S \backslash \{a\} \qquad \text{and} \qquad M = \max S \backslash \{a\} \in S \backslash \{a\}.$$

Letting $N = \max\{M, a\}$ and $n = \min\{n, a\}$, which both exist because X is totally ordered, it is easy to check that

$$n = \min S \qquad \text{and} \qquad N = \max S. \qquad \Box$$

This was relatively easy, though it did require induction. We could have used only $\#A = k$ in the inductive hypothesis.

Exercise 4 *Give an example of a partially ordered set X and a finite subset $A \subset X$ for which neither* $\min A$ *nor* $\max A$ *exist.*

Definition 3 *(least upper bound and greatest lower bound) An element $U_0 \in X$ is a* **least upper bound** *of a set $A \subset X$ if the following hold:*

1. *U_0 is an upper bound of A, and*

2. *For every upper bound U of A, we have $U_0 \leq U$.*

Similarly, ℓ_0 is a **least upper bound** *of A if*

1. *ℓ_0 is a lower bound of A, and*

2. *For every lower bound ℓ of A, we have $\ell \leq \ell_0$.*

Exercise 5 *Least upper bounds and greatest lower bounds (when they exist) are unique.*

In view of Exercise 5, we call any least upper bound U_0 of a set A *the* least upper bound of A or *the supremum* of A and write

$$U_0 = \sup A. \tag{1}$$

Similarly, we call any greatest lower bound ℓ_0 of A *the* greatest lower bound of A or *the infemum* of A and write

$$\ell_0 = \inf A.$$

In most, but not quite all, situations the **supremum** of A is synonymous with the least upper bound of A. Here is one important exception: If $X = \mathbb{R}$ and $A \subset \mathbb{R}$ is **not bounded above**, we can (and will) write

$$\sup A = \infty \qquad \text{or} \qquad \sup A = +\infty.$$

In this instance $\sup A$ does not denote the least upper bound of A because there is no (least) upper bound of A; the least upper bound of A does not exist. It is important to note that with this standard usage,

> **The supremum of any subset of $\mathbb{R}$ exists in $(-\infty, \infty]$.**

Thus, sometimes $\sup A$ is an **extended real number**. Recall that $(-\infty, \infty] = \mathbb{R} \cup \{\infty\}$ with

$$\begin{aligned}&x < \infty \text{ for every } x \in \mathbb{R},\\ &x + \infty = \infty \text{ for every } x \in \mathbb{R}, \text{ but}\\ &\infty \text{ has no additive inverse.}\end{aligned}$$

Sometimes, but very rarely, people write $\sup \phi = -\infty$. (Notice I said the supremum of **any** (!) subset of $\mathbb{R}$ exists.) If you say this, you're usually talking about the extended real number line $[-\infty, \infty]$ (or at least $[-\infty, \infty) = \{-\infty\} \cup \mathbb{R}$. There is no sensible meaning of $-\infty + \infty$.

Returning to more reasonable considerations, we have the following:

Definition 4 *X is said to be* **Dedekind complete** *if the following condition holds:*

> *Whenever $A \subset X$ is nonempty and bounded above, then A has a least upper bound.*

Exercise 6 *Give three fundamentally different and interesting examples of totally ordered sets which are* **not** *Dedekind complete.*

Recall the fundamental fact:

> **$\mathbb{R}$ is Dedekind complete.**

The condition

> Whenever $A \subset X$ is nonempty and bounded above, then A has a least upper bound.

is called the **least upper bound property**. We emphasize

> $\mathbb{R}$ **has the least upper bound property.**

Exercise 7 *If $A \subset \mathbb{R}$ is bounded below, then show that A has a greatest lower bound. Formulate a* **greatest lower bound property** *for partially ordered sets.*

In view of this discussion, we can restate our main result as

Theorem 2 *$\mathbb{Z}$ and $\mathbb{N}$ are Dedekind complete. If fact, $\mathbb{Z}$ and $\mathbb{N}$ satisfy the stronger condition:*

> *Whenever $A \subset X$ is nonempty and bounded above, then A has a maximum element.*

The condition

> Whenever $A \subset X$ is nonempty and bounded above, then A has a maximum element.

is called the **maximum element property**.[1]

2 Proof of the Theorem

Notice that since $\mathbb{N} \subset \mathbb{Z}$, we only need to consider $X = \mathbb{Z}$. In fact, the assertion holds for any subset $X \subset \mathbb{Z}$. In addition to Lemmas 1 and 2, will also use the following two results about the integers:

Lemma 3 *If $n_1, n_2 \in \mathbb{Z}$ with $n_1 < n_2$, then there is some (unique) $k \in \mathbb{N}$ with*

$$n_2 = n_1 + k.$$

Lemma 4 *Given $k \in \mathbb{Z}$, there is no integer $m \in \mathbb{Z}$ with $k < m < k + 1$.*

[1] Really I just made this up. I've never heard of "the maximum element property" before.

Proof of Theorems 1 and 2: Consider a set $A \subset \mathbb{Z}$ with $A \neq \phi$ and

$$a \leq U \in \mathbb{Z} \qquad \text{for every } a \in A.$$

Consider the set

$$B = \{m \in \mathbb{Z} : a \leq m \leq U \text{ for all } a \in A\}.$$

Notice that $U \in B \neq \phi$. Let $a_1 \in A$. If $a_1 = U$, then $a_1 = \max A$ by Lemma 1 part (a). Otherwise, $a_1 < U$ and we have by Lemma 3 some $k \in \mathbb{N}$ such that

$$a_1 < a_1 + 1 < a_1 + 2 < \cdots < a_1 + k = U.$$

Furthermore, since any element $n \in B$ satisfies $n \geq a_1$, we have

$$B \subset C = \{a_1, a_1 + 1, a_1 + 2, \ldots, a_1 + k = U\}.$$

This means $\#B \leq k + 1$ and B is a **finite set**. By Lemma 2, we know B has a minimum element

$$\min B = U_1 = a_1 + j \qquad \text{for some } j \in \{0, 1, 2, \ldots, k\}. \tag{2}$$

We claim that $U_1 = a_1 + j \in A$ so that

$$U_1 = a_1 + j = \max A. \tag{3}$$

Assume (BWOC) that $a_1 + j \in B \backslash A$. Then consider $a_1 + j - 1$. Notice that by Lemma 4 there is no integer m with

$$a_1 + j - 1 < m < a_1 + j.$$

In particular, there is no element $a \in A$ for which $a_1 + j - 1 < a \leq a_1 + j$. Therefore,

$$a \leq a_1 + j - 1 \leq U \qquad \text{for all } a \in A.$$

This implies $a_1 + j - 1 \in B$ contradicting the assertion of (2) that $\min B = a_1 + j$. We have established (3) which means $\mathbb{Z}$ has the maximum element property.

The approach of Exercise 7 should apply to show $\mathbb{Z}$ has the minimum element property as well, i.e., each nonempty subset of $\mathbb{Z}$ which is bounded below has a minimum element. $\square$