Math 4317, Final Exam: Analysis

1. (a) (10 points) Define the term *compact*.

   (b) (15 points) Prove or disprove: An arbitrary intersection of compact sets is compact.

Solution:

(a) A set K is compact if any open cover of K has a finite subcover.

(b) Proof: Let $\{K_\alpha\}$ be an arbitrary collection of compact sets, and denote their intersection by $A = \cap K_\alpha$.

Let $\mathcal{U} = \{U_\beta\}$ be any open cover of A. We need to show that $\mathcal{U}$ contains a finite subcover.

Since any intersection of closed sets is closed, we know $V = A^c$ is open. Thus, $\mathcal{U} \cup \{V\}$ is an open cover of the entire space. In particular, this will cover K_{α_0} for any particular α_0. Since K_{α_0} is compact, we obtain a finite subcover $U_{\beta_1}, \ldots, U_{\beta_k}, V$ of K_{α_0}. In particular, these open sets must cover $A \subset K_{\alpha_0}$. However, the particular set $V = A^c$ does not include any element of A, and it follows that the sets $U_{\beta_1}, \ldots, U_{\beta_k}$ cover A and constitute a finite subcover of $\mathcal{U}$ which covers A as we needed to find.

2. Define the following terms:

(a) (5 points) uniformly continuous

(b) (5 points) pointwise convergent

(c) (5 points) differentiable

(d) (5 points) uniformly convergent

Solution:

(a) A function $f : X \to \tilde{X}$ between metric spaces is uniformly continuous if given $\epsilon > 0$, there is some δ such that $d(x, \xi) < \delta$ implies $\tilde{d}(f(x), f(\xi)) < \epsilon$.

(b) A sequence of functions $f_1, f_2, \ldots : X \to \tilde{X}$ converges pointwise to a function $f : X \to \tilde{X}$ if for each fixed $x \in X$ we have

$$\lim_{j \to \infty} f_j(x) = f(x).$$

(c) A real valued function defined in some open ball about $x \in \mathbb{R}$ is differentiable if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists.

(d) A sequence of functions $f_1, f_2, \ldots : X \to \tilde{X}$ is uniformly convergent to $f : X \to \tilde{X}$ if for any $\epsilon > 0$, there is some N so that $j > N$ implies

$$\tilde{d}(f_j(x), f(x)) < \epsilon \quad \text{for all } x \in X.$$

3. Consider the sequence of functions $f_j : [0,1] \to \mathbb{R}$ given by $f_j(x) = x^j$ for $j = 1, 2, \ldots$. Evaluate the following assertions and justify your answers:

(a) (5 points) Each f_j is uniformly continuous.

(b) (5 points) The sequence $\{f_j\}$ converges pointwise.

(c) (5 points) Each f_j is differentiable.

(d) (5 points) The sequence $\{f_j\}$ converges uniformly.

(e) (5 points) The sequence $\{f_j\}$ converges in the norm given by

$$\|f\| = \left(\int_0^1 [f(x)]^2 \, dx \right)^{1/2}.$$

(You do not need to prove this is a norm.)

Solution:

(a) This is correct. Given $\epsilon > 0$, if $|x - y| < \epsilon/n$, then $|x^n - y^n| = |x - y||x^{n-1} + x^{n-2}y + \cdots + y^{n-1}| \le n|x - y| < \epsilon$.

(b) This is correct too. The sequence $f_j(x) \to 0$ for $x < 1$ and $f_j(1) \to 1$.

(c) This is correct: $(d/dx)x^n = nx^{n-1}$.

(d) The sequence does not converge uniformly. When $\epsilon_0 = 1/2$, then no matter which N you pick, there are, for every $j > N$, points $x < 1$ with $x^j > 1/2$.

(e) This is correct:

$$\int_0^1 x^{2n} \, dx = \frac{1}{2n+1} \to 0.$$

4. (a) (10 points) Give a correct statement of the mean value theorem.

(b) (15 points) Use the mean value theorem to prove: If $f \in C^0[a,b]$ and the derivatives $f' : (a,b) \to \mathbb{R}$ exists with $f' \equiv 0$, then there is some constant $c \in \mathbb{R}$ such that $f(x) = c$ for $x \in [a,b]$.

Solution:

(a) MVT: If $\phi \in C^0[a,b]$ and the derivative $\phi' : (a,b) \to \mathbb{R}$ exists, then there is some $\xi \in (a,b)$ such that

$$\phi'(\xi) = \frac{\phi(b) - \phi(a)}{b - a}.$$

(b) Consider the function $\phi = f$. This function satisfies the hypotheses of the mean value theorem on every interval $[\alpha, \beta] \subset [a,b]$, and $\phi' \equiv 0$. Consequently, for some $\xi \in (a,b)$,

$$\frac{f(\beta) - f(\alpha)}{\beta - \alpha} = \phi'(\xi) = 0.$$

Therefore, $\phi(x) = f(x) \equiv c$ is constant.

5. (a) (10 points) Define the Riemann integral of a function $f : [a, b] \to \mathbb{R}$.

(b) (15 points) Let $Q = \{1, 1/2, 1/3, 1/4, \ldots\}$ and let $f : [0, 1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 0, & x \in Q \\ 1, & x \notin Q. \end{cases}$$

Determine whether or not f is Riemann integrable. Justify your answer.

Solution:

(a) Given a function $f : [a, b] \to \mathbb{R}$, the number I is the integral of f over the interval $[a, b]$ if the following holds:

For any $\epsilon > 0$, there is some $\delta > 0$ such that

$$\left| \sum_{j=0}^{k-1} f(x_j^*)(x_{j+1} - x_j) - I \right| < \epsilon$$

whenever $\mathcal{P} : a = x_0 < x_1 < \cdots < x_k = b$ is a partitition of $[a, b]$ with

$$\|\mathcal{P}\| = \max_j (x_{j+1} - x_j) < \delta$$

and $x_j^* \in [x_j, x_{j+1}]$ for $j = 0, 1, \ldots, k - 1$.

(b) This function is Riemann integrable with Riemann integral 1. To see this, let $\epsilon > 0$ and take N with $1/N < \epsilon/4$. Then consider any partition $\mathcal{P}$ with $\|\mathcal{P}\| < \epsilon/(4N)$.

There are at most $2N$ partitition values which may intersect Q at $x = 1/n$ for $n < N$. Thus, the maximum sums of the lengths of these intervals is less than $\epsilon/2$. In addition, the sums of the lengths of all distinct intervals containing one of the points $x = 1/n$ for $n \geq N$ is at most $\epsilon/2$. Therefore, any Riemann sum associated with P must have value S with

$$1 - \epsilon < S < 1.$$

It follows that

$$\exists \quad \int_0^1 f(x)\, dx = 1.$$

6. (Bonus points) Define a sequence by $a_1 = 1/2$ and $a_{j+1} = 1 - a_j/2$ for $j = 1, 2, 3, \ldots$. Prove that

$$a_1 < a_3 < a_5 < \cdots$$

and

$$a_2 > a_4 > a_6 > \cdots .$$

Solution: Use the extended claim: For each odd j

$$0 < a_j < a_{j+2} < 2/3,$$

and for each even j

$$2/3 < a_{j+2} < a_j < 1.$$

These follow pretty easily by induction since $a_1 = 1/2$, $a_2 = 3/4$, and $a_{j+2} = 1 - (1 - a_j/2)/2 = 1/2 + a_j/4$.