Facts from Analysis

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I got a request to "help" with a self assessment of the undergraduate math program and, very roughly speaking, someone is interested in how much students in 4317 are learning. No one is specifically interested in you as an individual and how much you are learning, nor in me as an instructor mind you. What has been requested is some kind of anonymous reporting along the following lines (and I quote):

> Identify an exam where you ask your students to **prove a fact in analysis**? Once you identify that exam and problem...

The boldface, the punctuation, and the wording are verbatim. I would not make this stuff up. Perhaps the administrators should undertake a "self assessment" in regard to written expression. Nevertheless, this esteemed evaluator of higher learning goes on to say that I'm supposed to rate each solution on a scale from 0 (poor) to 5 (excellent) and then tally up the results and turn them in (without my name or any of your names).

Of course, one can ask if this is worth doing. But trying to be a nice guy, I agreed to try to do it. Maybe it won't be too onerous, and I figure it might be sort of entertaining. At some level the idea is not so bad. I suppose you would like to be able to "prove a fact in analysis," whatever that might be. Why not? There were two other listed categories where the same data is requested. Those are:

> Identify an exam...**sequences of functions**? ...

and

> Identify an exam...**continuity vs uniform continuity**? ...

I told the guy I wouldn't be able to do it until the end of the semester...so that should buy us some time.

I suggest the following: I'll write down some "facts from analysis" here, and if you prove one of them, I'll rate it and put down a score. I'll post separate documents containing problems related to the other two categories. The motivation for you is that I'll try to choose these things from among things you really ought to learn in an analysis course. Without further ado, here are a couple facts from analysis I think you should be able to handle. In fact, so this is not so egregious, I'll try to break them up in pieces like they might appear on an exam. Typically, you'll need to complete at least one "starter" and a "problem." Even on an exam in analysis, you should write things that can be understood.

1 $\mathbb{Q}$ is incomplete

Starter 1 *Let R be an ordered set.*

1. *Define what it means for a set to be* **bounded above** *in R.*
2. *Define what it means for an element in R to be the* **least upper bound** *of a set in R.*
3. *Explain what it means for an ordered ring to be Dedekind complete.*

Starter 2 *Give an informal definition of the rational numbers $\mathbb{Q}$.*

Let

$$A = \{x \in \mathbb{Q} : x^2 < 2\}.$$

Problem 1

1. *Show that A is nonempty and bounded above.*
2. *Show that if $x_0 \in A$, then there is some other element $x_1 \in A$ with $x_1 > x_0$.*
3. *Show that A does not contain any of its upper bounds, i.e., there is no maximum of the set A.*

Problem 2

1. *Show that if $x_0 \leq 0$, then x_0 is not an upper bound for A.*
2. *Show that if $x_0 \in \mathbb{Q}$ satisfies*
$$x_0 > 0 \qquad \text{and} \qquad x_0 \notin A,$$
then x_0 is an upper bound for A.
3. *Show that if $x_0 \in \mathbb{Q}$ is an upper bound for A, then there is some $x_1 \in \mathbb{Q}$ with $x_1 < x_0$ which is also an upper bound for A.*

Problem 3 *Explain why the two previous problems show $\mathbb{Q}$ is not Dedekind complete.*

2 Intermediate Value Theorem

Okay, here is another standard fact from analysis which you should be able to prove.

Starter 3 *Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be* **continuous at a point** $x_0 \in \mathbb{R}$.

Problem 4 *Show that the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ is continuous at $x_0 = 5$.*

Starter 4 *Let $A \subset \mathbb{R}$ and consider $f : A \to \mathbb{R}$.*

1. *Define what it means for a function $f : A \to \mathbb{R}$ to be* **continuous** *on A.*
2. *Show the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ is continuous on $\mathbb{R}$.*

Problem 5 *(intermediate value theorem) Let $a, b \in \mathbb{R}$ and consider a continuous function $f : [a, b] \to \mathbb{R}$. Show that if v is in the open interval with endpoints $f(a)$ and $f(b)$, then there is some $x_* \in (a, b)$ with $f(x^*) = v$.*

3 Continuity and Open Sets

Starter 5 *Define what it means for a subset of $\mathbb{R}$ to be* **open**.

Problem 6 *If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $v \in \mathbb{R}$, then show $\{x \in \mathbb{R} : f(x) < v\}$ s an open set.*

Problem 7 *If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $U \subset \mathbb{R}$ is an open set, show $f^{-1}(U)$ is an open set.*

4 Maximum Value Theorem

Problem 8 *If $a, b \in \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ is continuous, then there is some $x^* \in [a, b]$ such that*

$$f(x^*) \geq f(x) \qquad \text{for all } x \in [a, b].$$

Starter 6 *Define what it means for a set $K \subset \mathbb{R}$ to be* **compact**.

Problem 9 *If $K \subset \mathbb{R}$ is compact and $f : K \to \mathbb{R}$ is continuous, then there is some $x^* \in K$ such that*

$$f(x^*) \geq f(x) \qquad \text{for all } x \in K.$$

5 Continuous Linear Functionals

One more: This might also be considered a "fact in analysis."

Starter 7

1. *Define what it means for a set X to be a* **normed vector space***.*
2. *Define what it means for a function $L : X \to \mathbb{R}$ where X is a normed vector space to be* **linear***.*

Problem 10 *Show that if $L : X \to \mathbb{R}$ is linear and continuous at a single point $x_0 \in X$, then L is continuous at every point $x \in X$.*

Problem 11 *Show that if $L : X \to \mathbb{R}$ is continuous, then there is some M such that*

$$\|L(x)\| \leq M|L(x)| \qquad \text{for every } x \in X.$$