Math 4317, Exam 3: Analysis (extra practice)

1. (25 points) Define the terms *continuity* and *uniform continuity.*

   Prove or disprove: The function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 1/(1+x^2)$ is uniformly continuous.

Solution: Given a function $f : A \to \mathbb{R}^m$ with $A \subset \mathbb{R}^n$, we say that f is *continuous* if for any $x_0 \in A$ and any $\epsilon > 0$ there is some $\delta > 0$ depending on x and ϵ such that

$$x \in A \cap B_\delta(x_0) \qquad \Longrightarrow \qquad f(x) \in B_\epsilon(f(x_0)).$$

We say that f is *uniformly continuous* on A if given any $\epsilon > 0$, there is some δ depending on ϵ such that for every x and x_0 in A

$$|x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \epsilon.$$

Proof: Let $\epsilon > 0$. Note that given $x_0, x \in \mathbb{R}$,

$$\begin{aligned} |1/(1+x^2) - 1/(1+x_0^2)| &= |x^2 - x_0^2|/[(1+x^2)(1+x_0^2)] && (1)\\ &= |x - x_0||x + x_0|/[(1+x^2)(1+x_0^2)] && (2)\\ &\le |x - x_0|(|x| + |x_0|)/[(1+x^2)(1+x_0^2)]. && (3) \end{aligned}$$

When $x \ge 1$, we have $|x| \le x^2$ and $|x|/[(1+x^2)(1+x_0^2)] \le 1$. When $x < 1$, the same inequality holds. We also get the same inequality if we exchange x and x_0. Therefore, continuing from (3), we have

$$|1/(1+x^2) - 1/(1+x_0^2)| \le 2|x - x_0|.$$

That is, if $|x - x_0| < \epsilon/2$, then $|f(x) - f(x_0)| < \epsilon$.

2. (25 points) Define the distance between two functions using the uniform norm.

Prove or disprove: The uniform limit of uniformly continuous functions is continuous.

Solution: Given two functions $f : A \to \mathbb{R}^m$ and $g : A \to \mathbb{R}^m$ with $A \subset \mathbb{R}^n$, the distance between f and g is

$$\operatorname{dist}(f, g) = \sup\{|f(x) - g(x)| : x \in A\}.$$

Proof: Let f be the uniform limit of uniformly continuous functions f_j. That is, given $\epsilon > 0$, there is some N such that

$$j \geq N \qquad \Longrightarrow \qquad \operatorname{dist}(f_j, f) < \epsilon.$$

Let $\epsilon_0 > 0$. From the convergence above, there is some N such that $\operatorname{dist}(f_N, f) < \epsilon_0/3$. Since f_N is uniformly continuous, there is some $\delta > 0$ such that $|f_N(x) - f_N(x_0)| < \epsilon_0/3$ whenever x and x_0 are in A with $|x - x_0| < \delta$.

Thus, if $|x - x_0| < \delta$, then

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \epsilon_0. \end{aligned}$$

This shows, in fact, that f is uniformly continuous.

3. (25 points) (16I(a)) Define what it means for a sequence of functions to be *Cauchy* in the uniform norm.

Prove or disprove: The sequence of functions $f_j : [0, \infty) \to \mathbb{R}$ by $f_j(x) = x/j$ is Cauchy in the uniform norm.

Solution: $\{f_j\}$ is *Cauchy* if for any $\epsilon > 0$, there is some N such that when $j, k > N$ there holds $|f_j - f_k|_{L^\infty} < \epsilon$.

Disproof: $|f_j(x) = f_k(x)|_{L^\infty[0,\infty)} = \sup\{|1/j - 1/k|x : x \in [0, \infty)\} = +\infty$ if $j \neq k$. Thus, this sequence is not Cauchy in $C^0 \cap L^\infty$.