Equivalence Relations, Order and Cardinality

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There is a reasonable treatment of these topics in Gunning's text, but they are so important (and because my in-class presentation was somewhat deficient in certain respects) I've decided to provide an alternative treatment. I hope it helps. Equivalence relations and order relations, like all **relations on a set** X are formally realized as subsets of the **Cartesian product** or cross product $X \times X$ of the set with itself:

$$X \times X = \{(x, y) : x, y \in X\}.$$

An **equivalence relation** R is a subset of $X \times X$ which is reflexive, symmetric, and transitive. Formally, this means

(i reflexive) $(x, x) \in R$ for every $x \in X$,

(ii symmetric) $(x, y) \in R$ implies $(y, x) \in R$, and

(iii transitive) $(x, y), (y, z) \in R$ implies $(x, z) \in R$.

Informally (and more intuitively) we express the condition $(x, y) \in R$ using **binary relation notation**. For example, we may say

$$(x, y) \in R \qquad \Longleftrightarrow \qquad x \sim y$$

or

$$(x, y) \in R \qquad \Longleftrightarrow \qquad x \sim_R y$$

or

$$(x, y) \in R \qquad \Longleftrightarrow \qquad x \approx y \tag{1}$$

or

$$(x, y) \in R \qquad \Longleftrightarrow \qquad x \,\square\, y. \tag{2}$$

The symbol doesn't matter too much, but the intuitive nature of this informal device is enhanced by the use of a symbol that resembles an "=" sign, because (oddly enough) equality is the most familiar form of equivalence. So (1) is probably better than (2), but note that both expressions on the right are more suggestive than $(x, y) \in R$ which is technically more rigorous and informal. In any case, one can translate:

Exercise 1 *Write down the conditions for a relation to be reflexive, symmetric, and transitive in terms of the binary relation symbols "$\sim$" and "Ξ."*

Exercise 2 *Let $\mathcal{C}$ be a collection of sets. Show, with all the details, that*

$$R_{card} = \{(A, B) \in \mathcal{C} \times \mathcal{C} : \text{ there exists a bijection } \phi : A \to B\}$$

is an equivalence relations. (This is a very important exercise. And the relation is called **cardinality equivalence**.

For cardinality equivalence, an alternative to the usual binary relation notation is usually used:

$$(A, B) \in R_{card} \qquad \iff \qquad \#A = \#B. \tag{3}$$

In general, the **partition** or quotient associated with an equivalence relation R on a set X is given by

$$\{[x] : x \in X\}$$

where

$$[x] = \{y \in X : (x, y) \in R\} \tag{4}$$

is called the **equivalence class** of the element x.

Exercise 3 *The following property is crucial: Show*

$$[x] = [y] \qquad \iff \qquad [x] \cap [y] \neq \phi.$$

Note: The definition (4) only applies when one starts with, i.e., is given, an element $x \in X$. Consequently, there is no such thing as an empty equivalence class.

In the case of cardinality R_{card}, the equivalence class for a set A is usually denoted by $\#A$. Notice that this is consistent with the notation (3).

Order Relations

Like an equivalence relation, an order relation $\mathcal{O}$ on a set X is a subset of the Cartesian product $X \times X$. An **order relation** is reflexive, anti-symmetric, and transitive. In terms of binary relation notation, the symbol "$\le$" is very commonly used, but there are, of course, alternatives. For example, the conditions required of an order relation $\mathcal{O} \subset X \times X$ with

$$(x, y) \in \mathcal{O} \qquad \Longleftrightarrow \qquad x \prec y$$

may be expressed as follows:

(i reflexive) $x \prec x$ for every $x \in X$,

(ii anti-symmetric) $x \prec y$ and $y \prec x$ implies $x = y$, and

(iii transitive) $x \prec y$ and $y \prec z$ implies $x \prec z$.

An order relation is also called a **partial order**. If one has an order relation "$\prec$" on a set X with the additional property

(iv comparability) $x \prec y$ or $y \prec x$ for every $x, y \in X$,

then the relation is said to be a **total order** or linear order, and the set X is said to be **totally ordered**.

Exercise 4 *Reexpress the conditions required by an order relation $\mathcal{O}$ and a total order $\mathcal{T}$ in the formal notation of $\mathcal{O} \subset X \times X$ and $\mathcal{T} \subset X \times X$.*

Each ordinal is ordered, and the order on each ordinal is a total order.

$$0 = \phi \text{ has the trivial order } \mathcal{O}_0 = \phi.$$

$$1 = \{\phi\} \text{ has the pretty trivial order } \mathcal{O}_1 = \{(\phi, \phi)\} = \{(0, 0)\}.$$

$2 = \{0, 1\}$ starts to get interesting:

$$\mathcal{O}_2 = \{(0, 0), (0, 1), (1, 1)\} = \big\{(\phi, \phi), (\phi, \{\phi\}), (\{\phi\}, \{\phi\})\big\}.$$

Exercise 5 *Write down the ordering $\mathcal{O}_3$ on $3 = \{0, 1, 2\}$.*

Notice that for $(\phi, \phi) \in \mathcal{O}_3$, we have $\phi \subset \phi$. That is, $0 \subset 0$. The "next" element $(\phi, \{\phi\})$ has $\phi \subset \{\phi\}$ as well, so $0 \subset 1$. These two pairs/inequalities weren't so interesting, but eventually we find the pair

$$(1,2) = (\{\phi\}, \{\phi, \{\phi\}\}).$$

Here two, the smaller set $1 = \{\phi\}$ is a subset of the larger $2 = \{\phi, \{\phi\}\}$. You can check that

> **The ordering of each ordinal is by set inclusion.**[1]

Exercise 6 *Show the ordinal* $n = \{i \in \mathbb{N}_0 : i \in n\}$ *is given by*

$$n = \{0, 1, \ldots, n-1\}$$

and

$$\mathcal{O}_n = \{(i,j) \in n \times n : i \subset j\}.$$

This exercise should indicate how complicated and tedious it can be to verify all the properties of the finite ordinals commonly used. Even the formulation of the properties can be complicated and tedious. Here is one that I attempted to present (maybe not so successfully) in class:

Theorem 1 $\#(n+1) \neq \#n$ *for* $n = 0, 1, 2, 3, \ldots$.

Proof: I will try to prove this by induction. Formally, I want to show

$$A = \{n \in \mathbb{N}_0 : \#n \neq \#(n+1)\} = \mathbb{N}_0,$$

so the base case is: Show $0 \in A$ or $\#0 \neq \#1$. This amounts to showing there is no bijection $\phi : 0 \to 1$, that is, no bijection $\phi : \phi \to \{\phi\}$. Notice that the codomain/target $1 = \{\phi\}$ has an element in it. Therefore, such a "function" cannot be surjective because there is no $x \in \phi$ for which we can have $\phi(x) = \phi$.

> **Note:** I have used the same symbol for the empty set and the presumed bijection. In general this is bad practice, but I like to use ϕ for functions and especially bijections. Also using a different symbol for the empty set

[1] We'll describe below another related, but rather more subtle, property of ordinals due to, i.e., either invented by or discovered by, John von Neumann. You can think about what that might be, or at least be on the lookout for it.

is cumbersome. (The standard one in latex, $\emptyset$ is ugly and somewhat long to type.) So I just use the same symbol and **keep track in my mind** of which ϕ is which. We will have a significant opportunity for similar mental gymnastics below. Again, as with the informal notation for equivalence or order relations, no harm is done, as long as one can translate. Here is a translation of the little argument above using the symbol $\emptyset$ for the empty set:

We want to show $0 \in A$ or $\#0 \neq \#1$. This amounts to showing there is no bijection $\phi : 0 \to 1$, that is, no bijection $\phi : \emptyset \to \{\emptyset\}$. Notice that the codomain/target $1 = \{\emptyset\}$ has an element in it. Therefore, such a "function" cannot be surjective because there is no $x \in \emptyset$ for which we can have $\phi(x) = \emptyset$.

Exercise 7 *Give a fundamentally different proof that* $\#0 \neq \#1$.

Next, we undertake the inductive step. Here we assume $n \in A$ and we want to show $n+1 \in A$. To say $n \in A$ means

$$\text{There is } \textbf{no} \text{ bijection } \phi_0 : n \to n+1. \tag{5}$$

Now we assume $n+1 \notin A$, that is, there exists a bijection $\phi : n+1 \to n+2$. Here is where the proof starts to get tricky. There are two steps, and the second step is a bit trickier than the first. I know I can write

$$n+1 = n \cup \{n\} \tag{6}$$

(and I'll keep in my mind that $n = \{0, 1, \ldots, n-1\}$ when I think of the first n on the right in (6)). Since $n \subset n+1$ (think $\{0, 1, \ldots, n-1\} \subset \{0, 1, \ldots, n\}$) I can restrict ϕ to the subset:

$$\phi_1 = \phi\big|_n : n \to n+2. \tag{7}$$

The function ϕ_1 is still an injection, so it is a bijection onto its image. Thus, if

$$\phi_1(n) = n+1, \tag{8}$$

then I am done because I can take $\phi_0 = \phi_1$, that is $\phi_0 : n \to n+1$ by

$$\phi_0(j) = \phi_1(j)$$

and contradict (5).

This brings us to the last (tricky) case. I want to claim first that if

$$\phi_1(n) \neq n + 1, \tag{9}$$

then

$$\text{There is some } j \in n = \{0, 1, \ldots, n-1\} \text{ such that } \phi(j) = n + 1. \tag{10}$$

Just writing this claim and looking back at (8), I see some potential confusion. Let me pursue a little bit of an aside to address this potential confusion.

> Recall that generally, if $f : A \to B$ is a function, then we use f in two fundamentally different ways:
>
> $$f(x)$$
>
> denotes the element in B assigned by f to the element $x \in A$, but
>
> $$f(S) = \{f(x) : x \in S\}$$
>
> denotes a subset of B consisting of all elements $f(x)$ with x in a particular subset $S \subset A$. Thus, we have **images of elements** $f(x)$ and **images of sets** $f(S)$. Now if we look at (8), it may not be immediately clear which one we mean. In fact, $n \in n + 1$, so it is possible to think of n as an element. On the other hand, $n \notin n$, and the domain of the restriction ϕ_1 in (7) does not contain n as an element, so this interpretation does not make sense in (8); there is no ambiguity. In fact, we meant n as a set in (8), and we also meant $n + 1$ is a set $n + 1 = \{0, 1, \ldots, n\}$ on the right of (8).
>
> In (10) however, we mean j as an element and $n+1 \in n+2$ as an element. Again (I guess) we just need to keep track—but this ambiguity can be a bit disorienting.
>
> Perhaps we need a different notations for the application of a function ϕ to an element and the application of a function ϕ to a set., but I'm not going to attempt that now.

Let me see if I can establish the claim (10). Assume there is no such element j. Then $\phi^{-1}(n + 1) \notin n$. Again, the function $\phi^{-1} : n + 2 \to n + 1$ is defined on elements and that is the way we are intending to use it here. There is only one other element in the domain of ϕ other than the elements of n, since

$$n + 1 = n \cup \{n\}.$$

This means we must have $\phi^{-1}(n+1) = n$. That is, the last elements are in one-to-one correspondence

$$\phi(n) = n + 1$$

by

$$\phi : n + 1 = \{0, 1, \ldots, n\} \to n + 2 = \{0, 1, \ldots, n + 1\}.$$

But this means ϕ applied to n, as a set, must have image the reminder of the set

$$(n + 2)\backslash\{n + 1\} = n + 1.$$

That is,

$$\phi(n) = n + 1 \qquad \text{(as sets)}.$$

Consequently, we have (8) and a contradiction of (9). This establishes the existence of j (the element) in (10).

Finally, we want to define a bijection $\phi_0 : n \to n + 1$ and get a final contradiction of (5). I think we're in a position to do it. Let me write down the definition of my bijection and then we'll see if it makes sense. We define $\phi_0 : n \to n + 1$ by

$$\phi_0(\ell) = \begin{cases} \phi(\ell) & \text{if } \ell \neq j \\ \phi(n) & \text{if } \ell = j. \end{cases}$$

First of all, we want/need to show this function ϕ_0 is well-defined. Remember we have (10), and that is crucial. If $\ell \neq j$, then $\phi(\ell) \neq \phi(j) = n + 1$, which[2] means

$$\phi(\ell) \in (n + 2)\backslash\{n + 1\} = n + 1.$$

Therefore, the first case in our definition of ϕ_0 makes sense. Also, $\phi(n) \neq n+1 = \phi(j)$ for the same reason, i.e., because ϕ is a bijection, so

$$\phi(n) \in (n + 2)\backslash\{n + 1\} = n + 1. \tag{11}$$

We have shown ϕ_0 is well-defined.

Next we consider another restriction,

$$\phi_0\big|_{n\backslash\{j\}} = \phi\big|_{n\backslash\{j\}} : n\backslash\{j\} \to n + 1.$$

[2]This is because ϕ is a bijection.

This function is a bijection onto its image. The question is, "Can we determine the image of this restriction?" We can:

$$\begin{aligned}\phi\left(n\backslash\{j\}\right) &= \phi\left((n+1)\backslash\{j,n\}\right)\\ &= \phi(n+1)\backslash\{\phi(j),\phi(n)\}\\ &= (n+2)\backslash\{n+1,\phi(n)\}\\ &= (n+1)\backslash\{\phi(n)\}.\end{aligned}$$

Thus, the image is all of the set $n+1$ except the one element $\phi(n)$. Thus, our definition of ϕ_0 gives a surjective function.

It remains to show ϕ_0 is injective. If $\phi_0(\ell) = \phi_0(m)$ for two elements $\ell, m \in n\backslash\{j\}$, then $\phi(\ell) = \phi(m)$, so $\ell = m$ because ϕ is a bijection. The final case, if if $\phi_0(j) = \phi_0(\ell)$ for some $\ell \neq j$. But then $\phi(\ell) = \phi(n)$ so $\ell = n \notin n$, which is a contradiction because the domain of ϕ_0 is the set n. This establishes that ϕ_0 is injective, and we are done. □

This was a lot of work to prove Theorem 1, which is not a very exciting theorem.

Exercise 8 *On which page*[3] *of Principia Mathematica do Bertrand Russell and Alfred North Whitehead prove* $1 + 1 = 2$*?*

Set inclusion induces an order (a total order) on any ordinal **and** on any set of ordinals. The latter assertion is more general, since any ordinal is a set of ordinals. After much pain and turmoil (or without it) one obtains ordinal arithmetic, so that

$$1 + 1 = 2 \qquad \text{and} \qquad 2 + 2 = 4.$$

See Russell and Whitehead's three volume, two thousand page, set of books for the details. I will mention one more important observation/construction related to ordinals. John von Neumann viewed ordinals as the totally ordered sets α with the extended inclusion property that

$$s(\xi) = \{\eta \in \alpha : \eta < \xi\} = \xi \qquad \text{for all } \xi \in \alpha.$$

The set $s(\xi)$ is called an **initial segment**.

A function $\phi : A \to B$ of partially ordered sets is said to be **order preserving** if

$$\phi(x) \leq \phi(y) \qquad \text{whenever} \qquad x \leq y.$$

[3] Answer: Page 86 of volume II. (Volume I contained 719 pages of preliminaries.) The proof was followed by the remark: "The above proposition is occasionally useful. It is used at least three times..."

Exercise 9 *Show that*

$$R_{ord} = \{(A,B) : \text{ there exists an order preserving bijection } \phi : A \to B\}$$

is an equivalence relation on any collection of totally ordered sets.

Exercise 10 *Say ordinal addition is defined using disjoint unions so that*

$$\omega + \omega = \omega \coprod \omega = \omega_{red} \cup \omega_{blue}$$

with the usual orders on $\omega = \mathbb{N}_0$ *and* $m_{red} < n_{blue}$ *for all* m *and* n. *If this ordinal* $\omega + \omega$ *is denoted by* $\omega 2$, *then what is* $(\omega + 1) + (\omega + 2)$. *Hint:* $\omega + 1 = \{0, 1, 2, \ldots, \omega\}$ *and* $1 + \omega = \{1_{red}, 0, 1, 2, \ldots\} = \omega$, *so ordinal addition is not commutative in general when you get above the finite ordinals.*

Cardinality and cardinal arithmetic

As we know, cardinality gives an equivalence relation (quite easily) on any collection of sets. In particular, we have mentioned that

$$\#\omega = \aleph_0 \qquad \text{and} \qquad \#(\omega + 1) = \#\omega.$$

The **cardinality order relation** is rather more subtle than either the cardinality equivalence relation or the order relation of (sets of) ordinals. Nevertheless, it doesn't take too long to outline the subtle part. First of all,

$$\mathcal{O}_{card} = \{(A,B) : \text{ there exists an injection } \phi : A \to B\}$$

is an order relation on any collection of sets. The reflexive and transitive properties are easy to verify. The anti-symmetry is the Cantor-Bernstein theorem (also know as the Schröder Bernstein theorem). I won't say more about this here, as I've written extensively (and exhaustingly and exasperatingly) about this theorem elsewhere.

What I will say, is that the order relation $\mathcal{O}_{card}$ on a class of sets $\mathcal{C}$ induces an order relation on

$$\mathcal{C}_{card} = \{\#A : A \subset \mathcal{C}\}$$

the collection of all equivalence classes associated with equivalence of cardinality.

Exercise 11 *Show*

$$\{(\#A, \#B) : \text{ there exists an injection } \phi : A \to B\}$$

is a well-defined order relation on $\mathcal{C}_{card}$.

If one wishes to talk about **cardinal arithmetic** formally, it happens in $\mathcal{C}_{card}$.

Exercise 12 *Verify that*

$$\aleph_0 + \aleph_0 = \#\left(\aleph_0 \coprod \aleph_0\right) = \aleph_0$$

and that $\omega + \omega \neq \omega$.