Covid 0.2
Topological Continuity

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The objective of this document is to prompt the creation of a solid, beautiful, and elegant proof of the fact that two seemingly different definitions of **continuity** are equivalent when they are both applicable.

1 The ϵ-δ Definition

Given metric spaces X and Y with distance functions d_X and d_Y, we say $f : X \to Y$ is **continuous at a point** $x_0 \in X$ if the following condition holds:

> Given $\epsilon > 0$, there is some $\delta > 0$ such that
>
> $$d_X(x, x_0) < \delta \qquad \Longrightarrow \qquad d_Y(f(x), f(x_0)) < \epsilon. \tag{1}$$

2 The Topological Definition

Given topological spaces X and Y, we say a function $f : X \to Y$ is **continuous at a point** $x_0 \in X$ if the following condition holds:

> Given any open set V in Y with $f(x_0) \in V$, there is some open set U in X such that $x_0 \in U$ and
>
> $$f(U) = \{f(x) : x \in U\} \subset V. \tag{2}$$

3 Common Ground

Not every topological space is a metric space, but every metric space is a topological space. More precisely, a distance function d on a (metric) space X induces a topology called the **metric topology** as follows:

Exercise 1 *Let X be a metric space. Let $\mathcal{T}$ be the collection of all subsets U of X having the following property:*

> *For each $x_0 \in U$, there is some $r > 0$ such that*
> $$B_r(x_0) = \{x \in X : d(x, x_0) < r\} \subset U.$$

Show $\mathcal{T}$ is a **topology** *(set of open sets) making X a topological space. That is, you need to show $\phi \in \mathcal{T}$, $X \in \mathcal{T}$, and $\mathcal{T}$ is closed under finite intersections and arbitrary unions.*

Therefore, our definitions of **continuity at a point** both make sense for a function $f : X \to Y$ when X and Y are metric spaces considered with respect to the metric topology.

4 What to Show

Let X and Y be metric spaces with distance functions d_X and d_Y. One should be able to show the following:

1. If $f : X \to Y$ is continuous at $x_0 \in X$ according to the ϵ-δ definition, then f is continuous at x_0 with respect to the metric topologies on X and Y.

2. If $f : X \to Y$ is continuous at $x_0 \in X$ with respect to the metric topologies on X and Y, then f is continuous at x_0 according to the ϵ-δ definition.

3. If $f : X \to Y$ is continuous at every point $x_0 \in X$, then
$$f^{-1}(V) = \{x \in X : f(x) \in V\}$$
is open in X for every open set V in Y.

4. If $f : X \to Y$ and
$$f^{-1}(V) = \{x \in X : f(x) \in V\}$$
is open in X for every open set V in Y, then f is continuous at every point $x_0 \in X$.