Continuity, Compactness, and the Metric Topology of $\mathbb{R}$

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This is the material of a lecture. It's all arguably pretty important. It is important if you want to learn analysis.

1 Continuity

We recall that a function $f : X \to Y$, where X and Y are metric spaces with distance functions d_X and d_Y respectively, is **continuous at** $x_0 \in X$ if the following is satisfied:

For any $\epsilon > 0$, there is some $\delta > 0$ such that for $x \in X$,

$$d_X(x, x_0) < \delta \qquad \Longrightarrow \qquad d_Y(f(x), f(x_0)) < \epsilon.$$

A function is **continuous on** X if f is continuous at each point $x_0 \in X$.

In particular, if we take $f : E \to \mathbb{R}$ where $E \subset \mathbb{R}$ and we use the **Euclidean metric** given by the absolute value, $d(x, y) = |x - y|$ in both the domain and the codomain, then the continuity condition may be written as

For any $\epsilon > 0$, there is some $\delta > 0$ such that

$$\left.\begin{array}{c} |x - x_0| < \delta \\ x \in E \end{array}\right\} \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \epsilon.$$

One needs some strong "feel" for how this definition works. If the following examples are considered carefully, then maybe they can help one develop that "feel." Please pay close attention to each one.

1.1 First Example

$f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ is continuous. Let $x_0 \in \mathbb{R}$. Then

$$|x^2 - x_0^2| = |x - x_0||x + x_0| \leq |x - x_0|(|x - x_0| + 2|x_0|).$$

Therefore, given $\epsilon > 0$, if

$$|x - x_0| < \delta = \min\left\{\frac{\epsilon}{4|x_0|}, |x_0|\right\}, \tag{1}$$

then

$$|x^2 - x_0^2| < \frac{\epsilon}{4|x_0|}(|x_0| + 2|x_0|) = \frac{3\epsilon}{4} < \epsilon,$$

so it looks like we get continuity. Note that δ depends on x_0. We can indicate this by writing $\delta = \delta(x_0)$ or $\delta = \delta_{x_0}$.

1.2 Second Example

$g : (0, \infty) \to \mathbb{R}$ by $g(x) = 1/x$ is continuous. Let $x_0 \in (0, \infty)$. Then for $x \in (0, \infty)$,

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x - x_0|}{xx_0}.$$

Also, $|x| \geq |x_0| - |x - x_0|$ which we can also write as $x \geq x_0 - |x - x_0|$.

Therefore, given $\epsilon > 0$, if

$$|x - x_0| < \delta = \min\left\{\frac{\epsilon|x_0|^2}{4}, \frac{|x_0|}{4}\right\}, \tag{2}$$

then

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = |x - x_0| \frac{1}{xx_0} \leq \frac{\epsilon|x_0|^2}{4} \frac{1}{\left(|x_0| - \frac{|x_0|}{4}\right) x_0} = \frac{\epsilon|x_0|^2}{4} \frac{4}{3x_0^2} < \epsilon.$$

Again, we have shown continuity, and again δ depends on x_0.

1.3 Uniform Continuity

Roughly speaking, if the value of δ can be chosen without dependence on the point x_0, then a continuous function is **uniformly continuous**. Here is the precise definition:

A function $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is **uniformly continuous** on X if the following holds:

For any $\epsilon > 0$, there is some $\delta > 0$ such that for every $x, \xi \in X$

$$d_X(\xi, x) < \delta \qquad \Longrightarrow \qquad d_Y(f(\xi), f(x)) < \epsilon.$$

Exercise 1 *If $f : X \to Y$ is uniformly continuous, then f is continuous (at each point of X).*

Example of Nonuniform Continuity

$f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ is **not** uniformly continuous.
Proof: Assume (BWOC) there is some $\delta > 0$ such that

$$|\xi^2 - x^2| < 1 \qquad \text{whenever} \qquad |\xi - x| < \delta.$$

Consider $x = n$ and $\xi = n + \delta/2$ where $n \in \mathbb{N}$. Then

$$|\xi^2 - x^2| = \left| n\delta + \frac{\delta^2}{4} \right| > n\delta.$$

This means $n\delta < 1$ for δ fixed and $n \in \mathbb{N}$. In other words,

$$n < \frac{1}{\delta} \qquad \text{for all } n \in \mathbb{N}.$$

We have shown $1/\delta$ is an upper bound for $\mathbb{N}$ contradicting the Archimedean property of $\mathbb{R}$. $\square$

Example of Uniform Continuity

Let $a, b \in \mathbb{R}$ with $a < b$. Then $f : [a, b] \to \mathbb{R}$ by $f(x) = x^2$ is uniformly continuous. This gives us an opportunity to look back at our proof that $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ is continuous. We need to see if we can somehow find a value of δ which is smaller than all the values

$$\delta_{x_0} = \min\left\{ \frac{\epsilon}{4|x_0|}, |x_0| \right\}.$$

This consideration should lead us rather quickly to the disturbing realization that our argument for the continuity of x^2 on $\mathbb{R}$ was **wrong**, for of course, when $x_0 = 0$, the "number" δ_{x_0} we have written down is either zero, since the second number $|x_0| = 0$, or just nonsense. Either way, we have not written down a positive number δ in this case as required by the definition. We had better fix this up. It's not a very serious, much less fatal, error for our proof:

$$\delta_{x_0} = \min\left\{\frac{\epsilon}{4(|x_0|+1)}, |x_0|+1\right\}.$$

Now we are not dividing by zero and we won't have $\delta = 0$ either. With this choice

$$|x^2 - x_0^2| = |x - x_0||x + x_0| \le \frac{\epsilon}{4(|x_0|+1)}(|x_0| + 1 + |x_0|) < \epsilon.$$

So, now we have really proved that $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ is continuous.

Moreover, it makes sense to try to consider

$$\min\{\delta_{x_0} : x_0 \in [a, b]\} = \min\left\{\min\left\{\frac{\epsilon}{4(|x_0|+1)}, |x_0|+1\right\} : x_0 \in [a, b]\right\}.$$

In fact, if $x_0 \in [a, b]$, then $a \le x_0 \le b$, and it's relatively easy to see that $|x_0| \le \alpha = \max\{|a|, |b|\}$ which is positive and independent of x_0. Thus, we can take

$$\delta = \frac{\epsilon}{4\alpha} \qquad \text{independent of } \xi \text{ and } x$$

and have that if $\xi, x \in [a, b]$ with $|\xi - x| < \delta$, then

$$|\xi^2 - x^2| = |\xi - x||\xi + x| \le \frac{\epsilon}{4\alpha}(|\xi| + |x|) < \epsilon.$$

Exercise 2 *Why doesn't this "easier" proof of continuity work for $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$?*

A Second Example

Given $a > 0$, the function $g : [a, \infty) \to \mathbb{R}$ by $g(x) = 1/x$ is uniformly continuous.
Proof: Again we can look back at our proof on the larger domain and think about the value of $\delta = \delta_{x_0}$ we used there, namely,

$$\delta_{x_0} = \min\left\{\frac{\epsilon x_0^2}{4}, \frac{x_0}{4}\right\}.$$

This suggests

$$\min\{\delta_{x_0} : x_0 \geq a\} = \min\left\{\min\left\{\frac{\epsilon x_0^2}{4}, \frac{x_0}{4}\right\} : x_0 \geq a\right\} \geq \min\left\{\frac{\epsilon a^2}{4}, \frac{a}{4}\right\}.$$

In fact, setting $\delta = \min\{\epsilon a^2, a\}/4$, we have

$$|g(\xi) - g(x)| = \frac{|\xi - x|}{\xi x} \leq \frac{\epsilon a^2}{4} \frac{1}{a^2} < \epsilon.$$

Note: We could have taken simply $\delta = \epsilon a^2/2$. $\square$

Exercise 3 *$g : (0, \infty) \to \mathbb{R}$ by $g(x) = 1/x$ is* **not** *uniformly continuous.*

Solution: Assume (BWOC) there exists some $\delta > 0$ such that

$$\left.\begin{array}{c} \xi, x > 0 \\ |\xi - x| < \delta \end{array}\right\} \qquad \Longrightarrow \qquad |g(\xi) - g(x)| < 1.$$

Consider $\xi = 1/(2n)$ and $x = 1/n$ for $n \in \mathbb{N}$ with $n > 1/(2\delta)$. Then

$$|\xi - x| = \frac{1}{2n} < \delta.$$

But

$$|g(\xi) - g(x)| = n \qquad \text{for all large enough } n.$$

This time we have shown $n < 1$ so that $\mathbb{N}$ is strictly bounded above by 1, which is just nonsense. $\square$

2 A Theorem

Let $a, b \in \mathbb{R}$ with $a < b$.

Theorem 1 *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is uniformly continuous. That is a continuous real valued function on a closed and bounded interval is uniformly continuous.*

Proof: This is going to be a long drawn out proof with lots of discussion, so pull up a chair (and get out a pencil). Let $\epsilon > 0$. Recall that by continuity we have, for each $x \in [a, b]$ some $\delta_x > 0$ such that

$$\left.\begin{array}{c} |\xi - x| < \delta_x \\ \xi \in [a, b] \end{array}\right\} \qquad \Longrightarrow \qquad |f(\xi) - f(x)| < \epsilon. \tag{3}$$

This says there is an open interval $(x - \delta, x + \delta)$ where we can get some useful continuity condition as indicated in Figure 1. Actually, Figure 1 doesn't illustrate

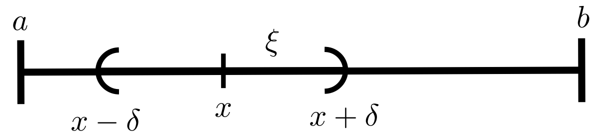

Figure 1: A δ ball with center x and radius δ.

much of anything about the continuity condition, but it illustrates the open interval in the domain that one gets from the continuity condition, and that is what we need to focus on for a while.

Recall that in a metric space X, given $r > 0$ and $x_0 \in X$, a set of the form

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\}$$

is called the **open ball** of radius r and center x_0. We're going to use this notation and terminology here, so our open interval becomes

$$(x - \delta_x, x + \delta_x) = B_{\delta_x}(x).$$

Now, we recall from the discussion/examples above that when we want uniform continuity, we basically want to find a minimum value for the tolerances δ we use:

$$\min\{\delta_x : x \in [a, b]\}.$$

In this case, that looks rather hopeless. First of all because there are infinitely many of these δ_x tolerances and, even more, because we have no idea how they depend on x or f or anything.

The following approach may not be obvious, but once I suggest it, I think you'll see it makes a lot of sense:

> *If we had only finitely many values δ_x, then we could easily take the minimum.*

Now which finitely many we want is also not so obvious, but we should have at least enough so that all the points $\xi \in [a, b]$ are included in their union. The following turns out to be (roughly) the right thing:
Claim: It only takes finitely many open balls $B_{\delta_x}(x)$ to **cover** $[a, b]$.

A collection of open sets $\{U_\alpha\}_{\alpha\in\Gamma}$ is an **open cover** (or just a cover) of a set A if

$$A \subset \bigcup_{\alpha\in\Gamma} U_\alpha.$$

Proof of the claim:[1] Assume (BWOC) that **no finite subcollection**

$$\{B_{\delta_{x_1}}(x_1), B_{\delta_{x_2}}(x_2), \ldots, B_{\delta_{x_k}}(x_k)\} \qquad \text{of the open cover} \qquad \{B_{\delta_x}(x)\}_{x\in[a,b]}$$

covers $[a, b]$. Okay, now we do some interval bisection.[2] If there is no finite subcover of $[a, b]$, then there is not finite subcover of one of the two intervals

$$\left[a, \frac{a+b}{2}\right] \qquad \text{and} \qquad \left[\frac{a+b}{2}, b\right]. \tag{4}$$

If these subintervals, each of which is clearly covered by $\{B_{\delta_x}(x)\}_{x\in[a,b]}$, both admitted finite subcovers, then we would have a finite subcover of the entire interval $[a, b]$. Let I_1 be one of the intervals in (4) with no finite subcover. Then we can repeat the argument: One closed half interval I_2 of I_1 has no finite subcover from $\{B_{\delta_x}(x)\}_{x\in[a,b]}$. Inductively, we get a sequence of nested closed intervals

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

[1]This will take a while and our "proof" will not quite work, so we'll have to come back and go over it again. But the proof contains many important ideas, so it's worth spending some time on and doing twice.

[2]I don't know who first had this idea, but it is the main idea in most proofs of what is called the **Heine-Borel theorem** which I'll discuss later.

with diameters

$$\operatorname{diam} I_j = \operatorname{diam}[a_j, b_j] = b_j - a_j = \frac{b-a}{2^j} \to 0.$$

Just in case, you need to be reminded: A sequence $\{x_j\}_{j=1}^{\infty}$ in a metric space X **converges** to $z \in X$ if for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that

$$j > N \qquad \Longrightarrow \qquad |x_j - z| < \epsilon.$$

Subclaim 1:

$$\bigcap_{j=1}^{\infty} I_j = \{z\} \qquad \text{is a singleton.}$$

Proof of subclaim 1: Let $I_j = [a_j, b_j]$ as above. Since

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

we know $a_1 \leq a_2 \leq a_3 \leq \cdots$. Also, $\{a_j\}_{j=1}^{\infty}$ is bounded above by b_1. Therefore, the limit

$$\lim_{j \nearrow \infty} a_j = \sup\{a_j\}_{j=1}^{\infty} = z \qquad \text{exists.}$$

Also, $z \in I_j = [a_j, b_j]$ for every j. Similarly,

$$\lim_{j \nearrow \infty} b_j = \inf\{b_j\}_{j=1}^{\infty} = \tilde{z} \qquad \text{exists with } \tilde{z} \leq b_j \text{ for every } j.$$

Therefore, $|z - \tilde{z}| \leq b_j - a_j$ for every j and $\zeta = |z - \tilde{z}|$ is a non-negative real number which satisfies $\zeta \leq \eta$ for every $\eta > 0$. This means $\zeta = 0$ and $z = \tilde{z}$. We have definitely shown

$$\{z\} \subset \bigcap_{j=1}^{\infty} I_j.$$

Exercise 4 *Show*

$$\bigcap_{j=1}^{\infty} I_j \subset \{z\}.$$

Recall there is some $\delta_z > 0$ such that

$$\left.\begin{array}{c} |\xi - z| < \delta_z \\ \xi \in [a, b] \end{array}\right\} \qquad \Longrightarrow \qquad |f(\xi) - f(z)| < \epsilon.$$

Subclaim 2: For j large enough $I_j \subset B_{\delta_z}(z)$.

Proof of subclaim 2: There is some $N \in \mathbb{N}$ such that

$$j > N \qquad \Longrightarrow \qquad \begin{cases} |b_j - a_j| < \delta_z/2 \\ |z - a_j| < \delta_z/2. \end{cases}$$

Thus, $x \in I_j = [a_j, b_j]$ implies

$$|x - z| \leq |x - a_j| + |a_j - z| \leq |b_j - a_j| + |a_j - z| < \delta_z. \qquad \square$$

Therefore $\{B_{\delta_z}(z)\}$ is a finite subcover of I_j (with one element). This contradicts the fact that I_j has no finite subcover. The contradiction establishes the claim. $\square$

Let's restate the claim we have just established like this: If $\{B_\delta(x)\}_{x\in[a,b]}$ is an open cover of $[a, b]$, then there exists a finite subcover

$$\mathcal{B} = \{B_\delta(x_1), B_\delta(x_2), \cdots, B_\delta(x_k)\}. \tag{5}$$

There are a few aspects of the foregoing proof of this claim it may be worthwhile to pause and point out. Let's start with a generalization which you should now be able to prove:

Exercise 5 *If $\{U_\alpha\}_{\alpha\in\Gamma}$ is any open cover of a set $A \subset \mathbb{R}$ which is closed and bounded, then there exists a finite subcover*

$$\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_k}\}.$$

This is essentially half (maybe more than half) of what is called the **Heine-Borel theorem** which I will come back to shortly. This property of having a finite subcover of any open cover does not always hold for closed and bounded sets in a topological space, but it provides the structure of a useful general concept:

Definition 1 *Given a topological space X, a set $K \subset X$ is said to be* **compact** *if every open cover of K contains a finite subcover.*

The Heine-Borel theorem generalizes our claim to general compact sets and to higher dimensional Euclidean space $\mathbb{R}^n$:

Theorem 2 *(Heine-Borel, Theorem 2.23 in Gunning) A subset $A \subset \mathbb{R}^n$ is compact if and only if A is closed and bounded.*

Some version of each of the subclaims above is also used to prove the Heine-Borel theorem. One of those in particular is worth stating:

Theorem 3 *(See Lemma 2.19 and Theorem 2.21 in Gunning) If $C_1 \supset C_2 \supset C_3 \supset \cdots$ is a nested sequence of nonempty closed sets in $\mathbb{R}^n$ with*

$$\operatorname{diam}(C_j) = \sup\{|x - \xi| : x, \xi \in C_j\} \to 0 \quad \textit{(as } j \to \infty\textit{)}$$

then

$$\cap_{j=1}^{\infty} C_j = \{z\} \qquad \textit{is a singleton.}$$

Returning to our claim (5), we note that for the purposes of the claim the radii δ may be taken as arbitrary positive numbers—not necessarily all the same so that $\delta = \delta_x$ and not necessarily arising as tolerances from any continuity condition. We never used any property of the radii related to continuity. We only used that each radius $\delta = \delta_x$ was positive. But assuming the δ_{x_j} do come from continuity, we can attempt to prove the assertion of the original theorem that f is uniformly continuous. Let's start by setting

$$\delta = \frac{1}{2} \min\{\delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_k}\}.$$

Then δ is a fixed positive number, which is the sort of tolerance we want for uniform continuity. We could leave off the factor of 1/2, but it doesn't hurt to take a δ which is a bit smaller to give us help with the triangle inequality. Let's see how that works out:

If $\xi, x \in [a, b]$ with $|\xi - x| < \delta$, then because $\mathcal{B}$ is a cover, there is some x_j with $x \in B_{\delta_{x_j}}(x_j)$. Also,

$$|\xi - x_j| \le |\xi - x| + |x - x_j| < \frac{1}{2}\,\delta_{x_j} + \delta_{x_j} = \frac{3}{2}\,\delta_{x_j}. \tag{6}$$

I would like, of course, to use continuity at x_j applied to ξ (and x):

$$|f(\xi) - f(x)| \le |f(\xi) - f(x_j)| + |f(x_j) - f(x)|. \tag{7}$$

But, as you can see, I've got a problem because (6) does not give me $\xi \in B_{\delta_{x_j}}(x_j)$. You can see there in (6), my factor of 1/2 is trying to help, but even if I took a smaller factor like 1/4 it still won't help enough. I've already used up the entire continuity tolerance δ_{x_j} in regard to x.

I'm going to have another problem too because my use of the triangle inequality in (7) gives me, according to the pointwise continuity condition (3)

$$|f(\xi) - f(x)| < |f(\xi) - f(x_j)| + \epsilon$$

which is not quite going to be good enough. That is, even if I could apply pointwise continuity at x_j with regard to ξ, I've also already used up all of my ϵ in application of pointwise continuity at x_j in regard to x.

These problems can be fixed, but we need to start back at the beginning of the proof. Note that (3) was essentially the first line of our proof. So we, in some sense, started with a fatal error in the first line. With what we know now, however, these errors are easily fixed up: First of all, for each $x \in [a,b]$, there is some δ_x such that

$$\left.\begin{array}{c}|\xi - x| < \delta_x \\ \xi \in [a,b]\end{array}\right\} \qquad \Longrightarrow \qquad |f(\xi) - f(x)| < \frac{\epsilon}{2}. \tag{8}$$

Compare to (3) and notice that I replaced ϵ with $\epsilon/2$. Is that okay? Does continuity allow me to do that?

Next, this gives me an open cover

$$\{B_{\delta_x/2}(x)\}_{x\in[a,b]}.$$

Again, I've replaced the radii δ_x with smaller (but still positive) radii $\delta_x/2$. I still get an open cover of $[a,b]$, and my argument above dividing the intervals in half over and over again still works and gives me a finite subcover

$$\mathcal{C} = \{B_{\delta_{x_1}/2}(x_1), B_{\delta_{x_2}/2}(x_2), \ldots, B_{\delta_{x_k}/2}(x_k)\}.$$

We'll keep our factor of 1/2 as before and set

$$\delta = \frac{1}{2}\min\{\delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_k}\}.$$

Now if $\xi, x \in [a,b]$ with $|\xi - x| < \delta$, then because $\mathcal{C}$ is a cover, there is some j with

$$x \in B_{\delta_{x_j}/2}(x_j).$$

Therefore,

$$|\xi - x_j| \le |\xi - x| + |x - x_j| < \frac{1}{2}\delta_{x_j} + \frac{1}{2}\delta_{x_j} = \delta_{x_j}.$$

Therefore, we can use the pointwise continuity condition (3) at x_j to get

$$|f(\xi) - f(x)| \le |f(\xi) - f(x_j)| + |f(x_j) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We have shown f is uniformly continuous. $\square$

3 Compactness

I'm going to restate the definition of compactness:

Definition 1 *Given a topological space X, a set $K \subset X$ is said to be* **compact** *if every open cover of K contains a finite subcover.*

Note that this applies in any metric space, any normed space, and any inner product space. In a certain sense, this is the "right" replacement for the requirement that a set be closed and bounded in spaces where being closed and bounded is not a strong enough requirement. Here are some relatively easy results you should be able to prove on your own. See also Theorems 2.21 and 2.22 of Gunning.

Theorem 4 *Any closed subset of a compact set is compact.*

Theorem 5 *Any compact subset of a metric space is closed and bounded.*

Theorem 6 *Any pointwise continuous function $f : K \to \mathbb{R}$ defined on a compact subset K of a metric space is uniformly continuous.*

Proof: For any $\epsilon > 0$, there is some $\delta = \delta_x > 0$ such that

$$d(\xi, x) < \delta_x \qquad \Longrightarrow \qquad |f(\xi) - f(x)| < \frac{\epsilon}{2}.$$

This means $\{B_{\delta_x/2}(x)\}_{x \in K}$ is an open cover of K. Because K is compact, there exists a finite subcover

$$\mathcal{B} = \{B_{\delta_{x_1}/2}(x_1), B_{\delta_{x_2}/2}(x_2), \ldots, B_{\delta_{x_k}/2}(x_k)\}.$$

Now, given any $\xi, x \in K$, there is some x_j such that

$$x \in B_{\delta_{x_j}/2}(x_j).$$

If

$$d(\xi, x) < \delta = \frac{1}{2} \min\{\delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_k}\},$$

then

$$d(\xi, x_j) \le d(\xi, x) + d(x, x_j) < \frac{\delta_{x_j}}{2} + \frac{\delta_{x_j}}{2} = \delta_{x_j}$$

and therefore,

$$|f(\xi) - f(x)| \le |f(\xi) - f(x_j)| + |f(x_j) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We have shown $f : K \to \mathbb{R}$ is uniformly continuous. $\square$

Notice that the last result, Theorem 6, applies to a continuous function $f : [a, b] \to \mathbb{R}$ as given in Theorem 1. But the proof seems much easier. The reason is that to apply Theorem 6 in the case of Theorem 1, one needs to know the interval $[a, b]$ is compact. This fact can be obtained, and is in some sense the main content of, the Heine-Borel theorem.

4 Topological Continuity

Finally, I will recall a definition stated in class but not yet recorded here in these notes:

Definition 2 *Given topological spaces X and Y, a function $f : X \to Y$ is* **continuous** *if*

$$f^{-1}(V) \text{ is open in } X \text{ whenever } V \text{ is open in } Y.$$

Exercise 6 *Let X and Y be metric spaces. Show that $f : X \to Y$ is continuous on X according to the ϵ-δ definition (metric continuous) if and only if $f : X \to Y$ is continuous according to the definition given above (topological continuity).*

Exercise 7 *Can you define a meaningful notion of* **continuity at a point** *for a function $f : X \to Y$ between topological spaces?*