Preliminary Remarks on Completeness

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February 7, 2020

Let us assume we have constructed the real numbers $\mathbb{R}$ from the rationals $\mathbb{Q}$. I am going to briefly review the basic concept involved in that construction which is **completeness** or more properly **Dedekind completeness**. The condition of completeness in this context means precisely this:

> Any **nonempty** subset of $\mathbb{R}$ which is **bounded above** has a **least upper bound**.

This is also a statement of the **least upper bound property** for $\mathbb{R}$. A totally ordered field like $\mathbb{R}$ is **complete** if it has **the least upper bound property**.

Let's back up a little bit and review some definitions.

1. A **ring** R is a set with two operations (addition denoted by "+" and multiplication denoted by " $\cdot$ " or by just writing elements next to each other) such that the following hold:

   (a) $(R, +)$ is an Abelian group.

   (b) $(R, \cdot)$ is an Abelian monoid.[1]

   (c) The distributive property holds: $a(b+c) = ab + ac$ for all $a, b, c \in R$.

2. A **field** is a ring F for which $(F^*, \cdot)$ is a group where $F^* = F\backslash\{0\}$ is the set of nonzero elements and (0 is the additive identity).

[1] In some contexts, both the requirement that R is Abelian under multiplication and that there is a multiplicative identity are relaxed. Thus, one speaks of a **commutative ring with unity** to describe our "Ring." If we wanted to relax the requirement that a ring is Abelian under multiplication, so we could consider the ring of square matrices for example, we can call such a set a **noncommutative ring.** In a noncommutative ring, one needs to require a left distributive property $(a+b)c = ac + bc$ as well. It's not so common to consider a ring with no multiplicative identity.

3. A **totally ordered ring** is a ring with an order relation induced by a designated set of positive numbers. Such an order relation is always a total order.

4. A **totally ordered field** is a field which is totally ordered as a ring.

The integers $\mathbb{Z}$ constitute a ring but not a field. The rationals $\mathbb{Q}$ form a field. Both are totally ordered.

If R is a ring and some finite sum of the multiplicative identity with itself gives the additive identity:

$$\sum_{j=1}^{p} 1 = 0,$$

then we say the field has **characteristic** p. Otherwise, the ring (or field considered as a ring) is said to have **characteristic** 0. In the latter case, there is an injective ring homomorphism $\phi : \mathbb{Z} \to R$ and the ring (or field) has a subring isomorphic to the ring of integers. This will be important below. The rings $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ all have characteristic 0.

In order to understand what it means for $\mathbb{R}$ to be complete, one must understand the definitions of **bounded above** and **least upper bound**. These are the following:

1. A subset A of an ordered set X is **bounded above** if there is some $x \in X$ such that $a \leq x$ for every $a \in A$. In this case, the number x is called an **upper bound** for A.

2. Given a set A which is bounded above in on ordered set X, an element $U \in X$ is a **least upper bound** for A if

   (a) U is an upper bound for A, and

   (b) If $x \in X$ is an upper bound for A, then $U \leq x$.

Remember, it's possible to have a totally ordered set which does not get its order relation from a designated set of positives. This just means the order relation (however it is designated as a subset of the Cartesian product of the set with itself) is a total order, i.e., for every pair of elements a and b, we have $a \leq b$ or $b \leq a$. But generally, when we talk about an "ordered ring" or an "ordered field," we mean specifically that the order comes from a designated set of positives, and in this case the order is always a total order.

In a totally ordered ring or field (like $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$) of characteristic 0 where there is always a copy of $\mathbb{Z}$, there is another property of interest:

> We say a totally ordered ring R of characteristic 0 has the **Archimedean property** or is **Archimedean** if for any $a \in R$, there is some $n \in \mathbb{Z}$ such that
> $$a < n.$$

There are a number of consequences of the Archimedean property which one can prove. Here are two important ones:

Lemma 1 *Let F be an Archimedean field.*

1. *If $a \in F$ satisfies $a > 0$, then there is some $n \in \mathbb{Z}$ such that*
$$0 < \frac{1}{n} < a.$$
2. *If $a \in F$, then there is some $n \in \mathbb{Z}$ such that $n < a$.*

Here, of course, when we say $n \in \mathbb{Z}$, we mean in the injected copy of $\mathbb{Z}$ determined by
$$n \mapsto \sum_{j=1}^{n} 1. \tag{1}$$

Exercise 1 *Show that in an Archimedean ring R that for any $a \in R$, there is some $n \in \mathbb{Z}$ such that $n < a$.*

Even in the totally ordered monoid $\mathbb{N}$ or the totally ordered bimonoid $\mathbb{N}_0$ which can be viewed (retrospectively) as containing injected copies of parts of $\mathbb{Z}$ by the restriction of the map (1) to $\mathbb{N}_0$, the Archimedean property we have formulated above makes sense and both may be considered Archimedean.

Summary

We have introduced three properties that may or may not apply to a totally ordered ring[2] of characteristic 0. These are being **complete** (i.e., having the least upper bound property), being a **field** (i.e., having nonzero elements form a multiplicative group), and being Archimedean (i.e., having arbitrarily large integers). I have not given an example of a non-Archimedean field. Of course, if a field (or ring) has

[2] And sometimes to a more general ordered set.

characteristic p, for example the finite field $\mathbb{Z}_p$ where p is prime, then that would be an example. But I have not given an interesting example of a non-Archimedean field of characteristic 0. I will leave it to you to look up such examples which include fields with intriguing names like the **hyperreal numbers** and the **surreal numbers**.

For our purposes, the following should be noted:

1. $\mathbb{Z}$ is complete and Archimedean—but not a field.

2. $\mathbb{Q}$ is a field and Archimedean—but not complete.

3. $\mathbb{R}$ is a complete Archimedean field. In fact, $\mathbb{R}$ is the **only** complete Archimedean field.

In addition, as we have mentioned above, the sets of natural numbers $\mathbb{N}_0$ and $\mathbb{N}$ also may be considered as complete and Archimedean. They are, of course, not fields.

As a final note, we mention that all of these sets have a property we have not yet formally defined,[3] but is important and will be defined soon. The sets $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are all **metric spaces**. In fact, $\mathbb{R}^n$ is also a metric space as is **any subset** of $\mathbb{R}^n$ for any n—which if you think about it includes all our previous examples and many more. For metric spaces there is a different notion of completeness which is called **metric completeness** or sometimes **Cauchy completeness**. Oddly enough, though it requires, in principle, more structure (a distance function or metric) to describe what it means to be metrically complete, the condition itself is weaker than being Dedekind complete (when both properties make sense). That is to say, a ring which is Dedekind complete is always metrically complete, but it is possible to have a ring which is metrically complete but not Dedekind complete. In summary, one can prove the following result:

Theorem 1

1. *If a totally ordered ring is Dedekind complete, then it is metrically complete.*

2. *If a totally ordered ring is metrically complete* **and** *Archimedean, then it is Dedekind complete.*

As a consequence (corollary) we can say $\mathbb{Z}$ is both metrically and Dedekind complete, and the field $\mathbb{Q}$ is neither Dedekind nor metrically complete. These assertions, of course, should all be much more meaningful when the definition of metric completeness is familiar.

[3] You should not let this slow you down from learning about it on your own.