Coroducts
and
The Axiom of Choice

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1 Introduction

This is a proposed solution of the exercise from my notes *Products and the Axiom of Choice*.

Exercise 1 *If the following assertion holds:*

> *Given any surjective function* $f : A \to B$*, there exists a function* $g : B \to A$ *such that* $f \circ g = \mathrm{id}_B$*.*

then the Axiom of Choice holds.

Of course, this came from the informal exercise stated in class which morphed into the claim

> Claim: Given any surjective function $f : A \to B$, there **is** a function $g : B \to A$ for which $f \circ g = \mathrm{id}_B$.

We proved this claim, but the proof required the Axiom of Choice. The discussion below proves (I think) that one needs the Axiom of Choice.

The proof uses, critically it seems, the construction of a **disjoint union**. Gunning discusses the disjoint union of two sets in §1.2 when he writes about cardinal addition. The notion of a disjoint union is pretty straightforward, and I will explain it presently. Since it is so central to my solution/proof below I will be pretty careful with it and use it quite formally—which, in this case, also makes the proof much clearer. Generally, I don't use disjoint unions that often, but I did mention one near the end of my solution for Exercise 6 from my discussion of the Cantor-Bernstein theorem (which is posted as the example of a proposal for a "hard" problem). In any case, a very formal construction of the disjoint union requires only the product of two sets, so no additional axiom is required. We do not need, for example an "Axiom of Disjoint Unions."

2 Coproducts

By a **coproduct** I simply mean a **disjoint union**.[1] The idea of a disjoint union is pretty straightforward: When you form the union of a family of sets $\{A_\alpha\}_{\alpha\in\Gamma}$ using the Axiom of Unions, you have the sets A_α in hand, so it makes sense that you should be able to keep track and "tag" each element $a \in \cup A_\alpha$ as it enters the union with the index α of the set A_α from whence it came. Thus, the element $a \in \cup A_\alpha$ becomes, if we are careful to put on tags when forming the union, a_α. When we keep track of the origin of each element in the union, we get a diffent set: If $A = \{0, 1\}$ and $B = \{1\}$, then $A \cup B = \{0, 1\}$, but the disjoint union is something like $\{0_A, 1_A, 1_B\}$ or if we write $A = A_1$ and $B = B_2$, the disjoint union might become $\{0_1, 1_1, 1_2\}$. In any case, it seems apparent, then, that we need a new name/symbol for the disjoint union, and our categorical friend Bourbaki makes the spendid suggestion $\coprod$. Thus, given a collection $\{A_\alpha\}_{\alpha\in\Gamma}$ we denote the **disjoint union** of the sets A_α by

$$\coprod_{\alpha\in\Gamma} A_\alpha.$$

Now this "tagging" procedure works pretty well most of the time, but the construction can be made more formal, and I want to make it more formal. Here is a way to do that: We form the product of $\cup A_\alpha$ and Γ. This uses the Axiom of Unions and the

[1] I must confess that my motivations for introducing the term *coproduct*, though they are several, are not exceedingly compelling. First of all, I read about coproducts on the wikipedia page as an alternative terminology for disjoint union, and the term was new to me. The best thing, and first primary motivation, was that there is a nice symbol $\coprod$ associated with the term (and existing in latex and for which I have never had any use), so it is always nice to find a new and useful symbol. There are various symbols (apparently) used to denote disjoint unions, and I had previously considered $\oplus$ and have probably even used that, but it does not seem to be a standard one, though apparently $\uplus$ and $\sqcup$ (used by Gunning) are. This particular symbol $\coprod$ comes from a particular backwater of mathematics called *category theory* for which I also have no use. Finally, the term as it appears in the title of this document follows nicely on the title *Products and the Axiom of Choice* which also provides a primary motivation for its use.

specification of products within the set of all functions from Γ into the union $\cup A_\alpha$. The result looks like this:

$$\left(\bigcup_{\alpha\in\Gamma} A_\alpha\right) \times \Gamma = \left\{(a,\alpha) : a \in \bigcup_{\alpha\in\Gamma} A_\alpha \text{ and } \alpha \in \Gamma\right\}.$$

Within this product we specify the disjoint union:

$$\coprod_{\alpha\in\Gamma} A_\alpha = \left\{(a,\alpha) \in \left(\bigcup_{\alpha\in\Gamma} A_\alpha\right) \times \Gamma : a \in A_\alpha\right\}.$$

Thus, our tags are realized by $a_\alpha = (a,\alpha)$. There's not so much difference here, but we're going to use a natural projection on a $\coprod A_\alpha$ below, and

$$(a,\alpha) \mapsto a$$

is a much more intuitive and clear symbolic representation of a projection than $a_\alpha \mapsto a$.

3 Proof of The Axiom of Choice

We recall that the Axiom of Choice concerns a product of sets

$$\prod_{\alpha\in\Gamma} A_\alpha. \tag{1}$$

The Axiom of Choice is the following:

> The product of a nonempty collection of nonempty sets is nonempty.

So we need an axiom to assert the product like that in (1) is nonempty.

3.1 A First Try—or The Setup

Now we are assuming an assertion which gives us the existence of a function g, and the Axiom of Choice asserts the existence of an element in a product—which we know is a certain function, so this looks promising. First of all, then, let's try to identify some domains and codomains so that the one-sided inverse g will be our element of the product. An element of the product (1), considered as a function, is a function p with

$$p : \Gamma \to \bigcup_{\alpha \in \Gamma} A_\alpha.$$

Thus, if we want to realize p as the function g, and we know $g : B \to A$, then we should set

$$B = \Gamma \qquad \text{and} \qquad A = \bigcup_{\alpha \in \Gamma} A_\alpha$$

and proceed to look for an appropriate function $f : A \to B$. The problem is, there is no obvious function

$$f : \bigcup_{\alpha \in \Gamma} A_\alpha \to \Gamma.$$

What we're saying is you want to take an element a of the union, and somehow coherently connect it back with an index for the union—and presumably the index of one of the sets A_α from which the element a came. But you don't know, in $\cup A_\alpha$ any particular set A_α from which the element a came. Given $a \in \cup A_\alpha$, you can look at

$$\{\alpha : a \in A_\alpha\},$$

and you know this set is nonempty, but choosing a particular element from it to form a function $f : \cup A_\alpha \to \Gamma$ would be tantamount to using the Axiom of Choice. And we don't want to use the Axiom of Choice to prove the Axiom of Choice. So we need to construct a function f in some other way; we need to do something different.

3.2 A Second Try

The way forward should now be, more or less, obvious. Given a collection of nonempty sets $\{A_\alpha\}_{\alpha \in \Gamma}$, let us consider a function

$$f : \coprod_{\alpha \in \Gamma} A_\alpha \to \Gamma.$$

In the coproduct

$$\coprod_{\alpha \in \Gamma} A_\alpha = \left\{ (a, \alpha) \in \left(\bigcup_{\alpha \in \Gamma} A_\alpha \right) \times \Gamma : a \in A_\alpha \right\}.$$

we have a natural projection $\pi : \coprod A_\alpha \to \Gamma$ by $(a, \alpha) \mapsto \alpha$. Thus, we can take $f = \pi$. We know, furthermore, that π is a surjective function as long as each of the sets A_α is nonempty.

> Let's check this last assertion carefully, just for good measure. If $\alpha \in \Gamma$, then there is a set A_α, and we know $A_\alpha \neq \phi$. Therefore, there is an element $a \in A_\alpha$ and, hence, an element $(a, \alpha) \in \coprod A_\alpha$. And for this element, we have $\pi(a, \alpha) = \alpha$. So, indeed $f = \pi$ is surjective.

Now, the assertion we are assuming says that any surjective function f has a right inverse. This means we know there is a function $g : \Gamma \to \coprod A_\alpha$ such that $f \circ g = \mathrm{id}_\Gamma$. This looks sort of promising, but we need to note that

$$g : \Gamma \to \coprod A_\alpha$$

is **not** our desired element p of the product $\prod A_\alpha$. The function g does not have the correct codomain in particular. Thus, we should do the obvious thing which is consider the composition $p = \pi \circ g$. In terms of a mapping diagram:

$$\Gamma \xrightarrow{g} \coprod_{\alpha \in \Gamma} A_\alpha \xrightarrow{\pi} \bigcup_{\alpha \in \Gamma} A_\alpha.$$

We have now obtained a function p with the correct domain and codomain. It remains to check that p is actually an element of the product. In order to do this, we need to show

$$p(\alpha) \in A_\alpha \qquad \text{for each } \alpha \in \Gamma.$$

Let's see: Given $\alpha \in \Gamma$,

$$p(\alpha) = \pi \circ g(\alpha)$$

and what we know is that $f \circ g(\alpha) = \alpha$. Let us proceed by way of contradiction:

> If we assume $p(\alpha) \notin A_\alpha$, then $g(\alpha) = (a, \alpha_0)$ for some element $a \in A_{\alpha_0}$. Since we know $p(\alpha) \notin A_\alpha$, it is clear that
>
> $$\alpha_0 \neq \alpha.$$
>
> On the other hand, $f \circ g(\alpha) = f(a, \alpha_0) = \alpha_0$ because $f = \pi$ was the projection. Therefore, since $f \circ g = \mathrm{id}_\Gamma$, we have shown
>
> $$\alpha = \alpha_0.$$
>
> This is a contradiction and (apparently) establishes the assertion.

4 Surjective Functions and Right Inverses: Summary

I believe I have shown that our lowly exercise/claim

> Every surjective function has a right inverse.

is equivalent to the Axiom of Choice.

I cannot say that I am 100% sure that my argument above is correct. I need to find an expert in set theory and logic and send it to him to check. But it looks okay to me. And the proof is kind of cool I think.