The Cantor-Bernstein Theorem*

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December 23, 2019

If A and B are sets, and you have functions $f : A \to B$ and $g : B \to A$ which are both injective, then there is a bijection $h : A \to B$.

To prove this, let's start with a "strategy" (which in mathematical proofs is often called an "ansatz"[1]):

We will construct the bijection $h : A \to B$ having the form

$$h(x) = \begin{cases} f(x) & \text{for } x \in F, \\ g^{-1}(x) & \text{for } x \in A \backslash F. \end{cases} \tag{1}$$

The set $F \subset A$ (where we use the values of f to define h) will need to be specified, and it should be the case that $A \backslash F \subset g(B)$ (so that $g^{-1}(x)$ makes sense when $x \in A \backslash F$.). This starting point certainly does have the virtue that it makes h look like a function constructed from f and g. What else could we use to construct h?

Let's try to write down some things we know about these two sets F and $A \backslash F$. First of all, there may be points in A where we **know** we'll need to use values from f. For example, if $x \in A \backslash g(B)$, then there is no (obvious) way to define $h(x)$ other than setting $h(x) = f(x)$. This means we need

$$F \supset A \backslash g(B).$$

*This is also called the Schröder Bernstein Theorem. In the tradition of many major results, this theorem was apparently first formulated and proved correctly by a mathematician whose name is not connected to the theorem at all, namely Richard Dedekind. In this same tradition, the result was never proved correctly by Schröder, though Schröder did variously, find, announce, and publish incorrect proofs. At least it was stated by Cantor and proved by Bernstein (who gave a second correct proof), after which Dedekind gave a third correct proof for good measure.

[1] Literally, "initial placement of a tool at a work piece" (in German), meaning an unjustified form or assumption which turns out to work.

On the other hand, maybe $f(A)$ misses some points in B, so there's no way we can make h a surjective map without using g^{-1}, the only other function from (at least part of) A to B in sight. So we need

$$A\backslash F \supset g(B\backslash f(A)).$$

Notice that the use of g^{-1} will allow us, in principle, to have any particular $b \in B$ in the image of h, as long as we can take $x = g(b) \in A\backslash F$ so that $g^{-1}(x) = b$. Of course, we don't expect to do this for every $b \in B$ unless g is already a bijection.

So we have two sets

$$A\backslash g(B) \qquad \text{and} \qquad g(B\backslash f(A)).$$

Since the second one is a subset of $g(B)$, these two sets are disjoint. That's good. Now, we might ask at this point

Are these two sets enough? That is, is it true that

$$[A\backslash g(B)] \cup g(B\backslash f(A)) = A\ ? \tag{2}$$

An explicit example here helps. Say $f : \mathbb{N}_0 \to \mathbb{N}_0$ by $f(n) = 2n$ and $g : \mathbb{N}_0 \to \mathbb{N}_0$ by $g(n) = 2n+1$. Then $g(B) = \{1, 3, 5, \ldots\}$, so we need $h(n) = f(n) = 2n$ for every even number n. Therefore, in this explicit example with $A = B = \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, we have

$$A\backslash g(B) = \mathbb{N}_0\backslash\{1, 3, 5, \ldots\} = \{0, 2, 4, 6, \ldots\}$$

the even numbers. On the other hand, $f(A) = f(\mathbb{N}_0) = \{0, 2, 4, 6, \ldots\}$ is also the even numbers, therefore

$$B\backslash f(A) = \mathbb{N}_0\backslash\{0, 2, 4, 6, \ldots\} = \{1, 3, 5, \ldots\}$$

is the odd numbers, and

$$g(B\backslash f(A)) = \{3, 7, 11, \ldots\} = \{4n + 3 : n \in \mathbb{N}_0\}. \tag{3}$$

As we proved, the sets $A\backslash g(B)$ and $g(B\backslash f(A))$ are disjoint, but we can also see that their union is not all of $\mathbb{N}_0$. The set $g(B\backslash f(A))$ consists of **some** odd numbers, but definitely not all of them. Thus, we cannot expect (2) to hold.

Let's look further at our explicit example to see if it suggests a way to proceed. Of course, it's easy to find a bijection between $A = \mathbb{N}_0$ and $B = \mathbb{N}_0$; just take the identity, but we want to ignore that and see what we can build with the given maps f and g. So far, we can consider all the even numbers in A assigned to (even) numbers in B with values given by f. It's convenient to represent this as

$$\begin{aligned} h : A &\to B \\ f : 0 &\mapsto 0 \\ 1 &\mapsto ? \\ f : 2 &\mapsto 4 \\ 3 &\mapsto ? \\ f : 4 &\mapsto 8 \\ 5 &\mapsto ? \\ f : 6 &\mapsto 12 \\ &\vdots \end{aligned}$$

You may also find even better "graphical" representations of how this map must work. We could also fill in

$$\begin{aligned} h : A &\to B \\ f : 0 &\mapsto 0 \\ 1 &\mapsto ? \\ f : 2 &\mapsto 4 \\ g^{-1} : 3 &\mapsto 1 \\ f : 4 &\mapsto 8 \\ 5 &\mapsto ? \\ f : 6 &\mapsto 12 \\ g^{-1} : 7 &\mapsto 3 \\ &\vdots \end{aligned}$$

but let's first concentrate on the images of the evens listed in the first list involving only values of f. Looking at the $1 \in A$ up there, we see there is a kind of potential conflict. Were we to try to assign $h(1)$ using $g^{-1}(1) = 0$, then we conflict with the assignment $h(0) = f(0) = 0$ which we already know must hold. This means we **must** use $h(1) = f(1) = 2$, assuming our overall strategy is going to work. That is, we need $1 \in F$. Notice that there doesn't seem to be any conflict with this assignment in our explicit example.

The general version of this observation is the following: If we have an element $x \in A \backslash g(B)$, for which we know we need $h(x) = f(x)$, then $g \circ f(x)$ must also be in F. That is,

$$F \supset g \circ f[A \backslash g(B)]. \tag{4}$$

This is probably a good time for a couple exercises.

Exercise 1 *Show that under the ansatz (1) one must have (4).*

Exercise 2 *What is the set $g \circ f[A \backslash g(B)]$ in our explicit example?*

The solution to Exercise 1 should make it clear that we cannot always expect the old set $A \backslash g(B)$ in F and the new set $g \circ f[A \backslash g(B)]$ to be disjoint. On the other hand, it is clear from our explicit example that this process can lead to the identification of new (previously unknown) points which must be in F for our construction to work. Furthermore, every time we add such new points, there are other potential conflicts leading to the same conclusion, namely more points in F. This leads to a sequence of sets

$$\begin{array}{l} A \backslash g(B) \\ g \circ f[A \backslash g(B)] \\ (g \circ f)^2[A \backslash g(B)] \\ (g \circ f)^3[A \backslash g(B)] \\ \qquad \vdots \end{array}$$

all of which must be subsets of F. Letting $(g \circ f)^0$ denote the identity mapping, we can write this as

$$F \supset \bigcup_{n \in \mathbb{N}_0} (g \circ f)^n[A \backslash g(B)]. \tag{5}$$

Given all this expansion of the known extent of F we should, for the sake of our construction, make sure we're not running into the supposed complementary set $g(B \backslash f(A))$. The following exercise is useful for that.

Exercise 3 *Show $g(B \backslash f(A)) = g(B) \backslash (g \circ f)(A)$.*

We've noted above that the first set $A \backslash g(B)$ does not intersect $g(B)$ and therefore does not intersect $g(B \backslash f(A)) \subset g(B)$. All of the other sets we've found in F are subsets of $g \circ f(A)$, so the more precise expression from the exercise says they do not intersect $g(B \backslash f(A))$ either.

A similar expansion takes place in the complement $A\backslash F$ starting with $g(B\backslash f(A))$. This works as follows: Say we take an element $x \in g(B\backslash f(A))$. In our explicit example, this could be $x = 3$ which you can see in the lower mapping representation a page or so back. We know that we need $h(x) = g^{-1}(x)$ on such an element. (In the example, $h(3) = 1$.) This means that the image of x under f in B can never be attained as a value of the injective function f. If we're going to get $f(x)$ in the image of h, the task will have to be accomplished by applying g^{-1} to $g \circ f(x)$. In our explicit example, it will be noted that $g \circ f(3) = 13$ which is not a point in $g(B\backslash f(A))$ as you can see by looking at (3). In the general case, the (potentially) new point in $A\backslash F$ is $g \circ f(x) \in g \circ f[g(B\backslash f(A))]$. Again, we get a sequence of sets

$$\begin{array}{l} g(B\backslash f(A)) \\ g \circ f[g(B\backslash f(A))] \\ (g \circ f)^2[g(B\backslash f(A))] \\ (g \circ f)^3[g(B\backslash f(A))] \\ \quad\vdots \end{array}$$

all of which must be in $A\backslash F$.

Exercise 4 *Show that if our strategy for this proof is going to work, then we must have*

$$A\backslash F \supset \bigcup_{k \in \mathbb{N}_0} (g \circ f)^k[g(B\backslash f(A))].$$

Exercise 5 *Show that for each $k, n \in \mathbb{N}_0$, the sets*

$$(g \circ f)^n[A\backslash g(B)] \qquad \text{and} \qquad (g \circ f)^k[g(B\backslash f(A))]$$

are disjoint so that

$$\bigcup_{n \in \mathbb{N}_0} (g \circ f)^n[A\backslash g(B)] \qquad \text{and} \qquad \bigcup_{k \in \mathbb{N}_0} (g \circ f)^k[g(B\backslash f(A))] \tag{6}$$

are disjoint.

We might like to know that the union of the two sets in (6) is all of A. This does not seem to be obvious, and I'll guess it's not true in general. Fortunately, we don't need to know this to prove the Cantor-Bernstein theorem. Let's set

$$F_0 = \bigcup_{n \in \mathbb{N}_0} (g \circ f)^n[A\backslash g(B)] \qquad \text{and} \qquad G_0 = \bigcup_{k \in \mathbb{N}_0} (g \circ f)^k[g(B\backslash f(A))].$$

We can then use the definition (1) directly without reference to G_0, namely we can just set $F = F_0$.

It is clear that h is a well-defined function with domain A and range contained in B. Furthermore, if $h(a) = h(x)$, then there are essentially three possibilities:
CASE I: $a, x \in F_0$.

In this case, $h(a) = f(a)$ and $h(x) = f(x)$. Since f is injective, we know $a = x$.
CASE II: $a \in F_0$ and $x \in A \backslash F_0$.

We will show that this case is vacuous, leading to a contradiction. That is, the values of $h(a) = f(a)$ and $h(x) = g^{-1}(x)$ can never be the same. To this end, note that $a = (g \circ f)^n(\alpha)$ for some $n \in \mathbb{N}_0$ and some $\alpha \in A \backslash g(B)$. Therefore,

$$x = (g \circ f)^{n+1}(\alpha).$$

This means $x \in F_0$ and contradicts the fact that $x \in A \backslash F_0$.
CASE III: $a, x \in A \backslash F_0$.

In this case, $g^{-1}(a) = g^{-1}(x)$, so $a = x$ simply by application of g to both sides.

We have shown that h is injective.

Finally, let $b \in B$. We can then consider $g(b) \in A$. There are two possibilities:
CASE I: $a = g(b) \in F_0$.

In this case, $a = (g \circ f)^n(\alpha)$ for some $n \in \mathbb{N}_0$ and some $\alpha \in A \backslash g(B)$. If $n = 0$, then we have an immediate contradiction because $a = \alpha$ with $a \in g(B)$ and $\alpha \in A \backslash g(B)$. Thus, we can assume $n > 0$. It follows that

$$b = g^{-1}[(g \circ f)^n(\alpha)] = f[(g \circ f)^{n-1}(\alpha)].$$

Since $\xi = (g \circ f)^{n-1}(\alpha) \in F_0$, we have found ξ for which $h(\xi) = f(\xi) = b$.
CASE II: $a = g(b) \in A \backslash F_0$.

Here $h(a) = g^{-1}(a) = b$.

We have shown that h is surjective. $\square$

Exercise 6 *Consider*

$$h_1(x) = \begin{cases} f(x) & \textit{for } x \in A \backslash G_0, \\ g^{-1}(x) & \textit{for } x \in G_0. \end{cases}$$

What happens if one tries to show h_1 is a bijection? What about in the explicit example?