Math 4317, Assignment 5B

§1.3 Vector Spaces

1. The row rank of a matrix and the column rank are the same.

§2.3 Topology

Let X be a topological space.

2. Let $A \subset X$. Then
$$\mathcal{T}_A = \{A \cap U : U \text{ is open in } X\}$$
is a topology on A. This is called the **subspace topology**. True or False: $I = (0, 1]$ is open in $[0, 1]$?

Connected Sets

Definition A topological space X is **connected** if whenever $X = U_1 \cup U_2$ with U_1 and U_2 nonempty open sets, then $U_1 \cap U_2 \neq \phi$.

3. Any interval $I \subset \mathbb{R}$ is connected.

Convexity in a Vector Space

Definition A subset C of a vector space V is **convex** if
$$\{(1-t)v + tw : t \in [0, 1]\} \subset C \qquad \text{whenever } v, w \in C.$$
The linear combination $(1-t)v + tw$ is called a **convex combination** of v and w.

4. Any convex subset of a vector space is connected.

A topological space X is **Hausdorff** if for any $x, y \in X$, there are disjoint open sets U_1 and U_2 with $x \in U_1$ and $y \in U_2$.

5. Every metric space is Hausdorff. A singleton $\{x\}$ in a Hausdorff space is closed.

Limit Points

Let $A \subset X$ and $x_0 \in X$. The point x_0 is said to be a **strict limit point** or **cluster point** of A if for every open set (neighborhood) U with $x_0 \in U$, one has
$$(A \backslash \{x_0\}) \cap U \neq \phi.$$
Let us denote the set of strict limit points of a set A by $\operatorname{clus} A$.

6. $\operatorname{clus} A \subset \overline{A}$ and $\overline{A} = A \cup \operatorname{clus} A$.

Definition (convergence of a sequence in a topological space) A sequence $\{x_j\}_{j=1}^{\infty}$ in a topological space X **converges** to $z \in X$ if for any open set $U \subset X$ wtih $z \in U$, there is some $N \in \mathbb{N}$ such that

$$j > N \qquad \Longrightarrow \qquad x_j \in U.$$

In this case, we write $x_j \to z$ or $\lim_{j\to\infty} x_j = z$.

7. Let X be a topological space.
   (a) If X is Hausdorff and $x_j \to z$ and $x_j \to x$, then $x = z$.
   (b) If $A \subset X$ and $x \in \operatorname{clus} A$, then there is a sequence $\{x_j\}_{j=1}^{\infty} \subset A$ with $\lim_{j\to\infty} x_j = x$.
   (c) It is possible (in a non-Hausdorff topological space) to have $x_j \to z$ and $x_j \to z$ with $x \neq z$.

8. Gunning §2.3 Group I Problem 3

9. Gunning §2.3 Group I Problem 4

10. Gunning §2.3 Group I Problem 5

§2.4 Compact Sets

11. (a) Any intersection of compact sets is compact.
    (b) True or False: If $K \subset X$ and K is a compact set, then $A \cap K$ is compact for every set $A \subset K$.

12. Prove the Bolzano-Weierstrass theorem: A set $E \subset \mathbb{R}^n$ is compact if and only if

$$\left.\begin{array}{c} A \subset E \\ \#E = \infty \end{array}\right\} \qquad \Longrightarrow \qquad E \cap \operatorname{clus} A \neq \phi.$$

13. The usual cantor set is obtained by removing the open middle third $(1/3, 2/3)$ from $[0,1]$, removing the open middle third from the remaining two intervals $[0,1/3]$ and $[2/3,1]$, and then (at each stage) removing the open middle third from each iterval from among the collection of disjoint closed intervals remaining. The (countable) intersection of all the (nested) closed intervals occuring during this procedure is called the **Cantor middle thirds set**. The set contains no interval and is closed.

    Write down a careful set construction of a general version of the cantor set in which, given a number $\alpha \in (0,1)$, an open middle interval of length $\alpha\ell$ is removed from each closed interval of length ℓ at each stage of the procedure. With $\alpha = 1/3$ you should get the Cantor middle thirds set.

    What is the length of the Cantor set you have constructed?

Monotone Functions (inverses)

14. Let I be an interval with $\inf I \in \mathbb{R}$ and $u : I \to \mathbb{R}$ a non-decreasing function.
    (a) Show there is a unique smallest interval $J \subset \mathbb{R}$ with $u(I) \subset J$.
    (b) Consider $v : J \to \mathbb{R}$ by

    $$v(y) = \inf\{x \in I : u(x) \geq y\}.$$

    Explain why v is well-defined and draw/produce illustrations indicating how v is defined especially when u is discontinuous. (Explain the relation between v and u_- and u_+.)
    (c) Show that v is non-decreasing and left continuous.
    (d) What happens if $\inf I = -\infty$?

15. Let I, $u : I \to \mathbb{R}$, and $v : J \to \mathbb{R}$ be defined as in the previous problem.
    (a) Characterize a (jump) discontinuity $y_0 \in J$ of V in terms of $v^{-1}(\{y_0\})$.
    (b) Show that $v \circ u(x) \leq x$ for all $x \in I$.
    (c) Characterize the equality condition $v \circ u(x) = x$ in terms of $v^{-1}(\{u(x)\})$.