Math 4317, Assignment 3B

§1.3 Vector Spaces

For this section (Problems 1, 2, and 3) let V and W be finite dimensional vector spaces over a field F. As you know, a function $T : V \to W$ is **linear** if

$$T(av + bw) = aT(v) + bT(w) \qquad \text{for all } a, b \in F \text{ and } v, w \in V.$$

The collection of all linear transformations from V to W is denoted by $\mathcal{L}(V \to W)$. Some authors use the notation $\mathcal{L}(V, W)$ for the same set.

1. Use the **basis theorem** (Gunning Chapter 1, §1.3, Theorem 1.7) to prove that every basis of V has the same number of elements. This number is called the **dimension** of V and is denoted $\dim V$.

2. Define the **quotient** vector space V/W and show

$$\dim V = \dim W + \dim V/W.$$

3. If $T \in \mathcal{L}(V \to W)$, then the following are equivalent:

   1. $T : V \to W$ is injective.
   2. $\ker(T) = \{v \in V : T(v) = \mathbf{0}_W\} = \{\mathbf{0}_V\}$.

§1.2 Groups, Rings, Fields

The following terminology is not universal, but the concepts are widely used and similar terminology is common.

A function (especially a linear function) $f : V \to F$ from a vector space V over a field F into the field F is called a **functional** (or sometimes just a **function** in contrast to an operator; see below).

A linear function $L : V \to W$ from one vector space V to another W (assuming they are both vector spaces over the same field F) is called a **linear transformation** or **operator**. The set of all linear transformations $L : V \to W$ is denoted $\mathcal{L}(V \to W)$. In this context, the images of elements $L(v)$ are often denoted Lv.

4. Show the set of linear transformations $\mathcal{L}(V \to W)$ is a vector space over the field F. (You will need to define operations constituting a group structure on $\mathcal{L}(V \to W)$ as well as a scaling $F \times \mathcal{L}(V \to W) \to \mathcal{L}(V \to W)$.

5. Show $\mathcal{L}(V \to W)$ is a ring with respect to composition.

§2.1 Normed Vector Spaces

For this section (Problems 6-13) V and W are normed vector spaces. This means, in particular, that we require V and W to be vector spaces over the field $F = \mathbb{R}$. As above, in this context, images $L(v)$ are often denoted Lv.

Consider $\| \cdot \| : \mathcal{L}(V \to W) \to [0, \infty]$ by

$$\|L\| = \sup_{\|v\| \neq 0} \frac{\|Lv\|}{\|v\|}. \tag{1}$$

This is called the **operator norm** on $\mathcal{L}(V \to W)$.

The set

$$C^0(V \to W) = \{L \in \mathcal{L}(V \to W) : \|L\| < \infty\}$$

is called the set of **continuous linear operators** from V to W. This same set is called the set of **bounded linear operators** (or transformations) and is sometimes denoted by $B(V \to W)$.

6. Find a linear operator $L : V \to W$ for some vector spaces V and W such that $\|L\| = \infty$.

7. Show $C^0(V \to W)$ is a normed vector space with norm given by (1).

8. Show that given $L \in C^0(V \to W)$, the following holds for each $v_0 \in V$:

   For each $\epsilon > 0$, there is some δ for which

   $$\|v - v_0\| < \delta \qquad \Longrightarrow \qquad \|Lv - Lv_0\| < \epsilon.$$

9. Given $L \in \mathcal{L}(V \to W)$ such that for each $v_0 \in V$ there holds:

   For each $\epsilon > 0$, there is some δ for which

   $$\|v - v_0\| < \delta \qquad \Longrightarrow \qquad \|Lv - Lv_0\| < \epsilon,$$

   show $L \in B(V \to W)$.

Inner Product Spaces: More Structure than a Normed Vector Space

An **inner product** on a real vector space V is a positive definite, symmetric, bilinear function

$$\langle \cdot\,, \cdot \rangle : V \times V \to \mathbb{R}.$$

These three properties, in detail, are the following:

(i positive definite) $\langle v, v\rangle \geq 0$ for all $v \in V$ and

$$\langle v, v\rangle = 0 \qquad \text{if and only if} \qquad v = \mathbf{0}.$$

(ii symmetric) $\langle v, w\rangle = \langle w, v\rangle$ for all $v, w \in V$.

(iii bilinear)

$$\langle av + bw, z\rangle = a\langle v, z\rangle + b\langle w, z\rangle \qquad \text{and} \qquad \langle v, aw + bz\rangle = a\langle v, w\rangle + b\langle v, z\rangle$$

for all $a, b \in \mathbb{R}$ and $v, w, z \in V$.

A real vector space with an inner product is called an **inner product space**.

10. Show $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^{n} x_j y_j$ defines an inner product on $\mathbb{R}^n$.

Given any real inner product space V, we set

$$\|v\| = \sqrt{\langle v, v\rangle} \qquad \text{for } v \in V. \tag{2}$$

11. Prove the Cauchy-Schwarz inequality

$$|\langle v, w\rangle| \leq \|v\|\|w\| \qquad \text{for all } v, w \in V$$

on any real inner product space.

12. Prove that $\| \cdot \| : V \times V \to [0, \infty)$ given by (2) is a norm. Thus, every inner product space is a normed space with the norm defined in (2) which is called the **inner product norm**.

13. If V is an inner product space with norm defined by (2), then show

$$\langle v, w\rangle = \frac{1}{4}\left(\|v + w\|^2 - \|v - w\|^2\right).$$

This is called the **polarization identity**, and it says that the inner product is determined completely by the inner product norm.

Open sets in $\mathbb{R}$ and ϵ-δ continuty

(We will use this in our study of monotone functions.)

A set $U \subset \mathbb{R}$ is **open** if for any $x \in U$, there is some $r > 0$ such that

$$(x - r, x + r) \subset U.$$

A set $A \subset \mathbb{R}$ is said to be **closed** $A^c = \mathbb{R} \backslash A$ is open.

14. (a) Show that an "open interval"

$$(a, b) = \{x \in \mathbb{R} : a < x < b\} \subset \mathbb{R}$$

is open.

(b) Show that **any** union of open sets is open. Hint: Let $\{U_\alpha\}_{\alpha \in \Gamma}$ be any collection of open sets in $\mathbb{R}$ (indexed by Γ), and show

$$\bigcup_{\alpha \in \Gamma} U_\alpha = \{x \in \mathbb{R} : x \in U_\alpha \text{ for some } \alpha \in \Gamma\}$$

is open.

(c) Show that a "closed interval"

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\} \subset \mathbb{R}$$

is closed.

Definition (ϵ-δ continuity) Given an open set $U \subset \mathbb{R}$ and a real valued function $u : U \to \mathbb{R}$, we say u is **continuous at** $x_0 \in U$ if the following condition holds:

For each $\epsilon > 0$, there is some $\delta > 0$ such that

$$|x - x_0| < \delta \qquad \Longrightarrow \qquad |u(x) - u(x_0)| < \epsilon.$$

The same function is said to be **continuous** on U (or just continuous) if u is continuous at every $x_0 \in U$. The set of all continuous real valued functions with domain U is denoted $C^0(U)$.

15. (a) Show $C^0(U)$ is a vector space (over the reals).

(b) Let $u : [a, b] \to \mathbb{R}$ be a real valued function defined on the **closed** interval $[a, b]$. Give a reasonable ϵ-δ definition of what it should mean for u to be **continuous at** $x_0 \in [a, b]$. (The point is to deal with the endpoints a and b.)

(c) Let $u : E \to \mathbb{R}$ be a real valued function defined on **any** set $E \subset \mathbb{R}$. Give a reasonable ϵ-δ definition of what it should mean for u to be **continous on** E.