Math 4317, Assignment 1B

§1.2 Groups, Rings, Fields

1. A **monoid** is a set with a binary operation which is associative and which contains an identity element. Show that the identity element in a monoid is unique.

2. Gunning §1.2 Group I Problem 2

3. Gunning §1.2 Group I Problem 3

4. A function $\phi : G_1 \to G_2$ from one group G_1 to another G_2 is a **homomorphism** if $\phi(ab) = \phi(a)\phi(b)$ for every $a, b \in G_1$. A bijective homomorphism is called a group **isomorphism**, and two groups with a group isomorphism between them are said to be **isomorphic groups**.

   (a) Show that the **kernel**, $\ker(\phi) = \{a \in G_1 : \phi(a) = e\} = \phi^{-1}(e)$ where e is the identity element in G_2, of a homomorphism and the **image**, $\text{im}(\phi) = \{\phi(a) : a \in G_1\} = \phi(G_1)$, of a homomorphism are subgroups of the groups G_1 and G_2 respectively.

   (b) If H is a subgroup of a group G, one can consider the **left cosets** of H given by
   $$aH = \{ah : h \in H\} \subset G$$
   and the **right cosets** $Ha = \{ha : h \in H\} \subset G$. A subgroup H is called **normal** if $aH = Ha$ for every $a \in G$. If H is a normal subgroup of G, then show the set of all (left) cosets $G/H = \{aH : a \in G\}$ with operation $(aH)(bH) = (ab)H$ is a group. This group G/H is called the **quotient group** of G by (the normal subgroup) H.

   (c) Show the kernel of a homomorphism is always a normal subgroup.

   (d) If $\phi : G_1 \to G_2$ is a homomorphism, then show $\text{im}(\phi)$ and $G_1/\ker(\phi)$ are isomorphic groups. This is called the **first homomorphism theorem**.

5. Gunning §1.2 Group I Problem 6

§1.1-2 Sets and Numbers

6. Show that $f : A \to B$ is injective if and only if there is a function $g : B \to A$ such that $g \circ f = \text{id}_A$.

7. Show that $f : A \to B$ is surjective if and only if there is a function $g : B \to A$ such that $f \circ g = \text{id}_B$.

8. Given an equivalence relation "$\sim$" on a set A and the condition $[a] \cap [b] \neq \phi$ where $[\xi] = \{x \in A : x \sim \xi\}$ denotes the equivalence class of $\xi \in A$, show $[a] = [b]$.

9. Given the ordinals $\omega = \mathbb{N}_0$, $\omega + 1 = \mathbb{N}_0 \cup \{\mathbb{N}_0\} = \omega \cup \{\omega\}$, and $\omega + 2 = \omega + 1 \cup \{\omega + 1\}$, show there is no order preserving bijection $\phi : \omega + 2 \to \omega + 1$.

10. Which group properties do $\mathbb{N}$ and $\mathbb{N}_0$ satisfy with respect to addition and multiplication?

11. Show $\mathbb{N}_0 \times \mathbb{N}_0 = \{(a, b) : a, b \in \mathbb{N}_0\}$ is a **bimonoid**, i.e., a monoid with respect to addition and a monoid with respect to multiplication if

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d)\\ (a, b)(c, d) &= (ac + bd, ad + bc).\end{aligned}$$

Monotone Functions

We consider again a non-decreasing function $u : I \to \mathbb{R}$ defined on an interval I. Some of the problems below may provide more sophisticated approaches to some of the earlier problems on monotone functions.

12. Assume $x \in (a, b) \subset I$. The greatest lower bound of $u((x, b))$ may be denoted by

$$u_+(x) = \inf u((x, b)).$$

The greatest lower bound of a set (which is bounded below) is called the **infemum** of the set. Similarly, using the term **supremum** for the least upper bound, we can write

$$u_-(x) = \sup u((a, x)).$$

(a) Show $u_-(x) \leq u(x) \leq u_+(x)$.

(b) Conclude $S(x) = u_+(x) - u_-(x)$ is a well-defined non-negative function on $(a, b) \subset I$. The value $S(x)$ is called the **jump of** u **at** x.

(c) Extend the function S to be reasonably defined at all points of I.

13. Assume $x \in I$.

(a) Show that if $S(x) = 0$, then for any $\epsilon > 0$, there is some $\delta > 0$ for which

$$|u(\xi) - u(x)| < \epsilon \qquad \text{for } \xi \in I \text{ with } |\xi - x| < \delta.$$

(b) Show that if for any $\epsilon > 0$, there is some $\delta > 0$ for which

$$|u(\xi) - u(x)| < \epsilon \qquad \text{for } \xi \in I \text{ with } |\xi - x| < \delta,$$

then $S(x) = 0$.

14. Assume $[a, b] \subset I$. For each $n \in \mathbb{N}$, set

$$E_n = \left\{ x \in [a, b] : S(x) \geq \frac{1}{n} \right\}.$$

(a) Show

$$\{x \in [a, b] : S(x) > 0\} = \bigcup_{n=1}^{\infty} E_n.$$

(b) Show that if $x_1, \ldots, x_k \in E_n$ with $x_1 < \cdots < x_k$, then

$$\sum_{j=1}^{k} S(x_j) \le u(b) - u(a) \qquad \text{and} \qquad k \le n[u(b) - u(a)].$$

(c) Show that $\{x \in [a, b] : S(x) > 0\}$ is countable.

15. Let I be any nonempty interval. There exist well-defined **extended real numbers**

$$\inf I \in [-\infty, \infty) \qquad \text{and} \qquad \sup I \in (-\infty, \infty].$$

(a) Show that there exists a sequence of closed subintervals $[a_j, b_j] \subset I$ such that

$$a_1 \ge a_2 \ge a_3 \ge \cdots \qquad \text{and} \qquad \lim_{j\to\infty} a_j = \inf I$$

and

$$b_1 \le b_2 \le b_3 \le \cdots \qquad \text{and} \qquad \lim_{j\to\infty} b_j = \sup I.$$

(b) Show $\{x \in I : S(x) > 0\}$ is countable.