Math 4317, Assignment 1A

§1.1 Sets and Numbers

1. Gunning §1.1 Group I Problem 2
2. Gunning §1.1 Group I Problem 4
3. Gunning §1.1 Group I Problem 5
4. Gunning §1.1 Group I Problem 6
5. Gunning §1.1 Group I Problem 7
6. Gunning §1.1 Group I Problem 8 and Group II Problem 9
7. Gunning §1.1 Group II Problem 10
8. Gunning §1.1 Group II Problem 11

Monotone Functions

Problems 9-13 assume familiarity with the **field properties**, **order properties**, and **completeness** of the real numbers $\mathbb{R}$. The properties of the real numbers as a **complete ordered field** will be discussed in more detail later; see Chapter 1 §2. Your familiarity with $\mathbb{R}$ should be adequate to make sense of these problems on **monotone functions**.

A function $u : E \to \mathbb{R}$ where $E \subset \mathbb{R}$ is **monotone** if *one* of the following conditions holds:

(I) $u(x) \leq u(y)$ for all $x, y \in E$ with $x < y$.

(D) $u(x) \geq u(y)$ for all $x, y \in E$ with $x < y$.

The function u is said to be (strictly) **increasing** if

$$u(x) < u(y) \text{ for all } x, y \in E \text{ with } x < y.$$

The function u is said to be (strictly) **decreasing** if

$$u(x) > u(y) \text{ for all } x, y \in E \text{ with } x < y.$$

9. Assume a monotone function u satisfies (I). This condition is also called **non-decreasing**.
   (a) Let $x_0 \in \mathbb{R}$ be fixed. Show the set

   $$V = \{u(x) : x \in E \text{ and } x > x_0\}$$

   is **bounded below** but not necessarily **bounded above**. Note: To show V is **bounded below** you need to show there is a real number ℓ such that $\ell \leq v$ for every $v \in V$. To show V is *not necessarily* bounded above means to give an explicit example where V is not bounded above, i.e., there is no real number U such that $v \leq U$ for every $v \in V$. The number ℓ is called a **lower bound**. The number U, were such a number to exist, is called an **upper bound**.

(b) The completeness of the real numbers implies that a *nonempty* set of real numbers which is bounded below has a **greatest lower bound**, that is, a real number ℓ_0 which is a lower bound such that $\ell_0 \geq \ell$ for every lower bound ℓ. Does the set V from the previous part of this problem necessarily have a greatest lower bound? Note: If your answer is "yes," then you should prove it. If your answer is "no," then you should give an example, i.e., counterexample.

(c) If the set V from the first part of this problem has a greatest lower bound, show the set of lower bounds for V,

$$A = \{\ell : \ell \leq v \text{ for all } v \in V\},$$

is bounded above.

(d) If the set V from the first part of this problem has a greatest lower bound ℓ_0, show the least upper bound U_0 of the set A from the previous part satisfies $U_0 \leq \ell_0$.

10. (intervals) A set $I \subset \mathbb{R}$ is an **interval** if the following condition holds:

Whenever we have $x, y \in I$ with $x < y$, then we must have

$$[x, y] = \{\xi \in \mathbb{R} : x \leq \xi \leq y\} \subset I.$$

Show that every interval has exactly one of the following ten forms:

$$\begin{aligned} &\phi \\ (-\infty, \infty) &= \mathbb{R} \\ (-\infty, b) &= \{x \in \mathbb{R} : x < b\} \\ (-\infty, b] &= \{x \in \mathbb{R} : x \leq b\} \\ (a, \infty) &= \{x \in \mathbb{R} : x > a\} \\ [a, \infty) &= \{x \in \mathbb{R} : x \geq a\} \\ (a, b) &= \{x \in \mathbb{R} : a < x < b\} \\ [a, b) &= \{x \in \mathbb{R} : a \leq x < b\} \\ (a, b] &= \{x \in \mathbb{R} : a < x \leq b\} \\ [a, b] &= \{x \in \mathbb{R} : a \leq x \leq b\} \end{aligned}$$

Hint: Either an interval is bounded below—or it is not. They key is to find the numbers a and/or b.

11. Assume $u : I \to \mathbb{R}$ is a monotone non-decreasing function defined on an interval I.

(a) If $x_0 \in (a, b) \subset I$, show "the" least upper bound U_0 of $u((-\infty, x_0))$ and "the" greatest lower bound ℓ_0 of $u((x_0, \infty))$ are unique real numbers such that $U_0 \leq \ell_0$.

(b) If the least upper bound U_0 of $u((-\infty, x_0))$ and the greatest lower bound ℓ_0 of $u((x_0, \infty))$ both exist, show that

$$U_0 \leq u(x_0) \leq \ell_0. \tag{1}$$

Defintion If $x_0 \in I$ and *at least one* of the inequalities in (1) is strict, we say x_0 is a **point of discontinuity** of the monotone non-decreasing function u. Note: This definition does not require that both numbers U_0 and ℓ_0 exist.

12. Assume $u : I \to \mathbb{R}$ is a monotone non-decreasing function defined on an interval I.
    (a) If $x_0 \in I$, when is it possible that neither the least upper bound U_0 of $u((-\infty, x_0))$ nor the greatest lower bound ℓ_0 of $u((x_0, \infty))$ exist?
    (b) If $x_0 \in I$ is a point of discontinuity of u, what are the possible relations between U_0, ℓ_0, and $u(x_0)$?
13. Assume $u : I \to \mathbb{R}$ is a monotone non-decreasing function defined on an interval I. Show the set of discontinuities of u is (at most) countable.

The Cantor-Bernstein Theorem

14. Notes on the Cantor-Bernstein theorem, Exercise 2
15. Notes on the Cantor-Bernstein theorem, Exercise 3