Math 4317, Exam 2 Name and section: ____________________

1. (a) (10 points) Define the term *compact*.

   (b) (15 points) Prove that a nonempty compact set in a metric space is bounded.

Solution:

(a) A set K is compact if any open cover of K has a finite subcover.

(b) Let K be a nonempty compact set. Since K is nonempty, we can take a point $x_0 \in K$.

Consider the open cover $\{B_j(x_0) : j = 1, 2, 3, \ldots\}$. (For each point $x \in K$ there is some j with $j > d(x_0, x)$. This means $x \in B_j(x_0)$.)

Since K is compact, we can find a compact subcover of $\{B_j(x_0) : j = 1, 2, 3, \ldots\}$. This means there is a ball of maximum radius j_0 in the subcover. This means $K \subset B_{j_0}(x_0)$.

Name and section: ________________

2. (a) (10 points) Define what it means for a function $f : X \to \tilde{X}$ to be *continuous at a point* $p_0 \in X$ where X and $\tilde{X}$ are metric spaces with distances d and $\tilde{d}$ respectively.

(b) (15 points) Consider $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^2.$$

Prove f is continuous at $x_0 = 2$. (Use the absolute value as a norm, i.e., the usual metric on $\mathbb{R}$, in both the domain and range.)

Solution:

(a) f is *continuous at a point* $p_0 \in X$ if for any $\epsilon > 0$, there is some $\delta > 0$ such that

$$d(p, p_0) < \delta \qquad \text{implies} \qquad \tilde{d}(f(p), f(p_0)) < \epsilon.$$

(b) Let $\epsilon > 0$. Set $\delta = \min\{1, \epsilon/5\}$. If

$$d(x, 2) = |x - 2| < \delta,$$

then $1 < x < 3$, so $3 < x + 2 < 5$, and $|x + 2| < 5$. Therefore,

$$|f(x) - f(2)| = |x^2 - 4| = |x - 2||x + 2| < (\epsilon/5) \cdot 5 = \epsilon.$$

This means that f is continuous at 2.

Name and section: ____________________

3. (a) (10 points) Define the term *connected*.
   (b) (15 points) Prove that the interval $[0,1]$ is a connected subset of $\mathbb{R}$.

Solution:

(a) A set A is connected if whenever U_1 and U_2 are disjoint open sets such that $A \subset U_1 \cup U_2$, then one of the two sets $U_1 \cap A$ or $U_2 \cap A$ is empty.

(b) To see that $[0,1]$ is a connected subset of $\mathbb{R}$, assume there are two disjoint open sets U_1 and U_2 with $[0,1] \subset U_1 \cup U_2$, $[0,1] \cap U_1 \neq \phi$, and $[0,1] \cap U_2 \neq \phi$. We now seek a contradiction.

Under these assumptions $x = 0$ is in one of the sets $[0,1] \cap U_1$ or $[0,1] \cap U_2$. Without loss of generality, let us assume $0 \in [0,1] \cap U_1$. Since U_1 is open, there is some $\epsilon > 0$ for which $[0,\epsilon) \subset U_1$. We may assume $\epsilon \leq 1$.

Thus, the set $E = \{\epsilon : [0,\epsilon) \subset [0,1] \cap U_1\}$ is a nonempty set in $\mathbb{R}$ which is bounded above (by 1). By the least upper bound property, we know E has a least upper bound $0 < t \leq 1$. It follows also that

$$t \in E \cap U_1. \tag{1}$$

To see this, note that either $t \in U_1$ or $t \in U_2$. If $t \in U_2$, then there is some $r > 0$ such that $(t-r, t] \subset (0,1) \cap U_2$. In particular, the point $t - r/2 \in U_2 \backslash U_1$ which contradicts the definition of t since $t - r/4 < t$. Thus, $t \in U_1$, and if $0 \leq x < t$, then $x \in [0, t - (t-x)/2) \subset [0,1] \cap U_1$ by the definitions of t and E. That is, $[0,t) \subset [0,1] \cap U_1$, or $t \in E$.

It follows immediately from (1) that $[0,t] \subset U_1$. If $t < 1$, there is some $r > 0$ for which $[t, t+r) \subset (0,1) \cap U_1$, and we have an immediate contradiction of the definition of t. Therefore, $t = 1$ and $[0,1] \subset U_1$. This contradicts the assumption that $[0,1] \cap U_2 \neq \phi$.

An alternative definition of *connected* is that A and ϕ are the only sets that are both open and closed relative to the subspace A.

A proof that $[0,1]$ is connected based on this definition is as follows: Assume U is both open and closed in $[0,1]$. Then 0 must either be in U or the (open) complement V of U. Without loss of generality, let's say $0 \in U$. Then again, there is some interval $[0,\epsilon) \subset U$ with $0 < \epsilon \leq 1$. If we can take $\epsilon = 1$, then we can argue that $1 \in U$ as well, since U is closed. Thus, $U = [0,1]$ and $V = \phi$.

Alternatively, if there is some $v \in V$, then V is a nonempty set which is bounded below, and we can consider $v_0 = \inf V$ by the greatest lower bound property. Again, since V is closed, we know $v_0 \in V$. On the other hand, $v_0 > \epsilon$. Therefore, there is some interval $(v_0 - \delta, v_0] \subset V$, and this contradicts the definition of v_0.

Each of the proofs above can be simplified a little bit. Let's take the first one: Start, as before, with $0 \in [0,1] \cap U_1$. Since we know $C_2 = [0,1] \cap U_2$ is nonempty and bounded below, we can set $v_0 = \inf C_2$ by the greatest lower bound property. Since U_1 is open, we know there is some interval $[0, \epsilon) \subset U_1$ with $0 < \epsilon < 1$. Therefore, $v_0 > 0$. Using the same reasoning, we can see that $v_0 < 1$.

Now, if $v_0 \in U_1$, then there is some interval $[v_0, v_0 + \epsilon) \subset C_1 = [0, 1] \cap U_1$. This contradicts the definition of v_0 as the *greatest* lower bound. Therefore, $v_0 \in U_2$. But U_2 is also open, so this means there is some $\delta > 0$ such that $(v_0 - \delta, v_0] \subset C_2$, and this contradicts the fact that v_0 is a lower bound.

Name and section: ____________________

4. (25 points) Label each of the following assertions as "true" or "false." If a statement is true, you do not need to explain why, but if a statement is false, give a counterexample. (Counterexamples need to be correct and carefully given, but you do not need to prove anything about them.)

   Some of the assertions below involve the *Cantor set*. We will learn more about the Cantor set later in the course, but all you need to know for this problem is that the Cantor set $\mathcal{C}$ is a closed subset of the unit interval $[0,1]$.

   (a) A closed and bounded set is compact.

   (b) The Cantor set is compact.

   (c) The inverse image of a connected set under a continuous function is connected.

   (d) The inverse image of an open set under a continuous function is open.

   (e) Given a continuous function $f : \mathcal{C} \to \mathbb{R}$ (with domain the Cantor set), there is some number M and some $x_0 \in \mathcal{C}$ with $f(x_0) = M \geq f(x)$ for every $x \in \mathcal{C}$.

   (f) Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of continuous real valued functions on a *non-compact* domain. If f_j converges uniformly to a function f, then f must be continuous.

Solution:

(a) A closed and bounded set is compact.

False: $(0,1)$ is closed and bounded in the *space* $(0,1)$, but is not compact.

(b) The Cantor set is compact.

True.

(c) The inverse image of a connected set under a continuous function is connected.

False: Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Then $f^{-1}(1,4) = (-2,-1) \cup (1,2)$ is disconnected.

(d) The inverse image of an open set under a continuous function is open.

True.

(e) Given a continuous function $f : \mathcal{C} \to \mathbb{R}$ (with domain the Cantor set), there is some number M and some $x_0 \in \mathcal{C}$ with $f(x_0) = M \geq f(x)$ for every $x \in \mathcal{C}$.

True.

(f) Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of continuous real valued functions on a *non-compact* domain. If f_j converges uniformly to a function f, then f must be continuous.

True.