Math 4317, Exam 1 Name and section: ____________

1. **(i)** State precisely the definition of a function. (Hint: Given two sets X and Y,...)

(ii) Given a function $f : X \to Y$, and a subset $B \subset Y$, define $f^{-1}(B)$.

(iii) Given a function $f : X \to Y$, and a subset $A \subset X$, define $f(A)$.

(iv) Given a function $f : X \to Y$ and $A \subset X$, show that

$$f^{-1}(f(A)) \supset A.$$

(v) Give an example to show the inclusion of part (iv) may be proper.

(25 points) **Solution:**

(i) A *function* is a rule or correspondence which assigns to each $x \in X$ a unique $y \in Y$.

(ii) $f^{-1}(B) = \{x \in X : f(x) \in B\}$.

(iii) $f(A) = \{f(x) : x \in A\}$.

(iv) If $x \in A$, then $f(x) \in f(A) \subset Y$. This means $x \in f^{-1}(B)$ where $B = f(X)$, by definition (ii).

(v) Let $f : \{1, 2\} \to \{0\}$ by $f(x) = 0$ for all x. Let $A = \{1\}$. Then $f^{-1}(f(A)) = \{1, 2\} \supsetneq A$.

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2. (25 points) Define *bounded above.*

Define *upper bound.*

Define *supremum.*

Solution:

1. Given a set A of real numbers, we say A is *bounded above* if there is a real number M with $M \geq x$ for every $x \in A$.
2. Given a set A of real numbers which is bounded above, we say a real number M is an *upper bound* for A if $x \leq M$ for every $x \in A$.
3. Given a nonempty set A of real numbers which is bounded above, a number M is the *supremum of* A (or the *least upper bound* of A), if M is an upper bound for A and if $B \in \mathbb{R}$ is any other upper bound for A, then $B \geq M$.

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3. **(i)** State precisely the least upper bound property for $\mathbb{R}$.

Let $A = \{\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots\}$.

(ii) Give a recursion formula relating the elements in A.

(iii) Does the least upper bound property apply to A?

(iv) Find $\sup A$ as an extended real number.

(25 points) **Solution:**

(i) A nonempty subset of $\mathbb{R}$ which is bounded above has a least upper bound in $\mathbb{R}$.

(ii) $a_{j+1} = \sqrt{2 + a_j}$.

(iii) We show that $a_j < 2$ for all j by induction. First, $a_1 = \sqrt{2} < 2$ since $2 < 4$. Next, if $a_j < 2$, then $a_{j+1} = \sqrt{2 + a_j} < \sqrt{2+2} = 2$.

(iv) The sequence $\{a_j\}$ is increasing. (This we will prove by induction below.) We showed in part (iii) that the sequence is bounded above, therefore it has a limit which is $\sup A = x$. Taking the limit on both sides of the recursion relation, we find $x = \sqrt{2+x}$. Thus, x is a positive solution of the quadratic equation $x^2 - x - 2 = (x-2)(x+1) = 0$. Therefore, $x = 2$.

Proof that the sequence is increasing: $a_1 < a_2$. That is clear. If $a_\ell < a_{\ell+1}$, then $a_{\ell+2} = \sqrt{2 + a_{\ell+1}} > \sqrt{2 + a_\ell} = a_{\ell+1}$.

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4. **(i)** Define what it means for a set in a metric space to be *open*.

(ii) For this question, let us define a *closed* set to be one whose complement is open. Using your definition from part (i), show that an arbitrary intersection of closed sets is closed.

(25 points) **Solution:**

(i) A set U in a metric space X is *open* if whenever $x \in U$, there is some $r > 0$ such that

$$B_r(x) = \{\xi \in X : d(\xi, x) < r\} \subset U$$

where $d : X \times X \to [0, \infty)$ is the distance function for X.

(ii) Let C_α be a closed set in a metric space X for every α in some indexing set Γ. If $x \in (\cap C_\alpha)^c = \cup C_\alpha^c$, then there is some α_0 such that x is in the open set $U = C_{\alpha_0}^c$. This means there is some $r > 0$ with

$$B_r(x) \subset U \subset (\cap C_\alpha)^c.$$

This means, $(\cap C_\alpha)^c$ is open, and that means $\cap C_\alpha$ is closed. $\square$